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Dynamic updating approximations approach to multi-granulation interval-valued hesitant fuzzy information systems with time-evolving attributes

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ABSTRACT

Since data is furious growth and rapid alteration, it is completely imperative to monitor and update real-time data promptly. There is no denying that calculating approximations by means of classical approach is pretty time-consuming for an information system with attribute sets varying constantly. Whereas, dynamic updating approximations method takes full advantage of previous knowledge instead of calculating from scratch, which saves a large amount of time. Enlightened by this idea, our work focuses on researching mechanisms of dynamic updating approximations caused by the variation of attributes in multi-granulation interval-valued hesitant fuzzy information system (MG-IVHFIS). To begin with, the average dominance relation which reduces the restriction of universal dominance relation in reality is recommended, then an average dominance rough set based on this relation is established in MG-IVHFIS. Additionally, we study four mechanisms for updating approximations from the perspective of optimism and pessimism in dynamic MG-IVHFIS when some attributes are removed or inserted, and improve corresponding dynamic algorithms. Furthermore, we test ten datasets from UCI and design contrastive experiments to assess dynamic and classical algorithms. In terms of computational efficiency, experimental results show that the dynamic method clearly precedes the classical method for handling with dynamic attribute sets in MG-IVHFIS.

1. Introduction

As science and technology advance, a variety of data are more readily available. Even though large amounts of data are boosting at an unprecedented speed, there are still quite a few missing, ambiguous and imprecise data. Rough set (RS) theory is of great essence in reference to powerful tools for disposing of the uncertainty. The RS theory brought up by Pawlak [1,2] is an extension of classical set. The RS centers on the lower and upper approximations and depicts uncertain knowledge through known knowledge. For purpose of fitting into the development of our era, plentiful improved models based on RS come into being, such as neighborhood rough set [3], double-quantitative rough set [4], fuzzy rough set [5], dominance-based rough set [6], rough set on two universes [7]. It has become an emerging academic hot spot in some fields of machine learning [8], data mining [9], pattern recognition [10] and so on.

Granular computing [11–14] is featured with expanding the dimension of thought from one angle and probing into information from multiple angles. During the period of 1996 to 1997, the concept of granular computing was proposed by Zadeh [15] for the first time. From the perspective of granular computation, an equivalence relation in the classical RS can be considered as a granularity, meanwhile a partition of equivalence relation over the domain can be deemed as a granularity space. Applying granular computation to the RS theory, a novel model referred as multi-granulation rough set (MGRS) was presented by Qian et al. [16]. MGRS digs out information from diverse granularities, which provides a more profound comprehension of hidden knowledge. As a consequence, many researchers [17–21] set out to make a study of some topics regarding MGRS. Xu et al. [22] raised a generalized multi-granulation rough set, introduced supporting characteristic function and discussed some important properties. Zhang et al. [23] extended two types of multi-granulation hesitant fuzzy rough sets model and presented related concepts of rough measure. The study [24] listed theoretical frameworks and important research ideas for a variety of multi-granularity data analysis methods.

Information system (IS) is a fundamental way to express knowledge in classical RS theory, which consists of objects and

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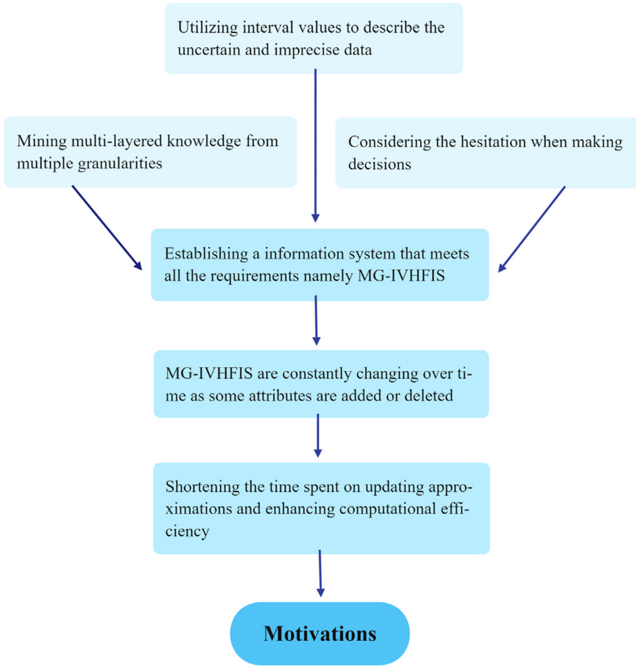


Fig. 1. Motivations of our work.

attributes over the universe and is exhibited in the shape of two dimensional table. In classical IS, there exists a binary relation, namely the equivalence relation. Nevertheless, the great mass of problems dissatisfy with the equivalence relation in practical application. Subsequently, according to specific forms of data that are collected by us, scholars have expanded the classical IS into manifold information systems, and conducted quite a lot of researches of RS on dissimilar information systems. If the domain of values for each attribute satisfies a partial ordered relation, an IS with this sort of relation is known as an ordered information system. Xu [25] introduced different ordered information systems in his book, such as set-valued ordered information system and intuitionistic fuzzy ordered information system. When interval values are substituted for definite single values, an interval-valued information system (IVIS) is generated, which better reflects the uncertainty of knowledge. Some researchers chose the study with IVIS as background. Leung et al. [26] formulated a knowledge discovery framework to analyze the IVIS. In combination with dominance relation, Qian et al. [27] extracted dominance rules in an interval ordered information system. Gong et al. [28] integrated classical RS with interval-valued fuzzy set, and studied RS in the interval-valued fuzzy information system. Huang et al. [29] made use of extended information entropy to measure the uncertainty in interval-valued intuitionistic fuzzy information system. A fuzzy dominance relation in the light of fuzzy dominance degree was proposed by Yan and Dai [30] in interval ordered information system. The literature [31] extended the RS to the MGRS in IVIS. When the number of interval values in the domain of values for all attributes is greater than one, the interval-valued hesitant fuzzy information system (IVHFIS) creates. Three new partial ordered relations were discussed and the definition of hesitant fuzzy information entropy was given by Lu and Yu [32] in IVHFIS. For one side, aiming to delve multi-faceted, deep-seated knowledge, for another, taking into account people's vacillation and hesitation adequately in the face of choice, the research background of our work is MG-IVHFIS.

The arrival of information age allows people to have access to the latest data from moment to moment. As new data enters, outdated and redundant data should be removed from earlier IS. Consequently, updating approximations in time guarantees the timeliness of knowledge in a time-evolving IS. In recent years, incremental knowledge discovery [33–35] has become an important tool for processing dynamic data sets. It updates new IS on the basis of original knowledge without recalculating entire contents. If an IS changes over time, one of the following three situations may occur: First, the object set varies while the rest remain the same. Second, the attribute set varies while the rest remain the same. Third, only the attribute values vary. To analyze these three cases, lots of scholars have come up with relevant dynamic updating algorithms in different information systems. In allusion to variations of object set, Luo et al. [36] studied two incremental algorithms for updating the approximations in set-valued information system. An incremental approach for updating approximations of dominance-based rough sets approach was developed by Li et al. [37]. Yu and Xu [38] worked on incremental approaches updating approximations with dynamic data sets in interval-valued decision system. In the incomplete interval-valued decision information system, the article [39] defined a multi-threshold tolerance relation and explored several static and dynamic algorithms for solving approximations. In allusion to variations of attribute set, a kind of dominance matrix was introduced to update dominating and dominated sets by Li et al. [40]. Based on the similarity-based rough set, Zhang et al. [41] investigated incremental approaches in IVIS. Yu et al. [42] proposed two dynamic computing rough approximations approaches for interval-valued ordered information system. In allusion to variations of attribute values, an incremental approach for maintaining approximations of dominance-based rough sets approach was recommended by Li [43]. The essay [44] mentioned the changing mechanisms of the attribute values and fuzzy equivalence relations, then advanced two corresponding incremental algorithms of fuzzy rough set. Chen et al. [45,46] presented the principles of dynamically updating approximations in the incomplete ordered decision systems and a new incremental method for updating approximations of variable precision rough set. Apart from that, some researchers have linked incremental knowledge discovery with attribute reduction and accomplished involved projects [47,48]. In recent times, several innovative topics regarding to updating approximations have been investigated by scholars [49–52]. For instance, Hu and Li [50] contrived a dynamic framework in a neighborhood multigranulation space. The literature [49] proffered the updating mechanisms of dynamic objects for double-quantitative decision-theoretic rough set. Matrix-based incremental updating approximations technique in multi-granulation rough set was devised by Xu et al. [20] when two-dimensional variation occurred simultaneously. Consistent with many scholars, this paper researches mechanisms of dynamic updating approximations resulting from the alteration of attribute set, so as to reduce calculation time.

With the diversification of data form, incremental algorithms are proposed for diverse types of information systems. So far, scholars have studied lots of topics about dynamic updating approximations, but these existing incremental algorithms are not appropriate for MG-IVHFIS. As Fig. 1 shows, there are other limitations. Inspired by above motivations, we research dynamic updating approximations approach to multi-granulation interval-valued hesitant fuzzy information systems with time-evolving attributes. In this paper, our main work is summed up as follows: (1) a new relation is constructed named the average dominance relation, which improves the original dominance relation. In terms of average dominance relation, we present a new model of rough set in MG-IVHFIS. (2) Four incremental mechanisms are explored when multiple attributes are inserted into or removed from a MG-IVHFIS, and the corresponding dynamic algorithms are contrived. (3) We perform some comparative experiments on ten data sets. Experimental results have confirmed that these proposed dynamic algorithms are superior to classical algorithms.

The arrangement of remanent paper is as following statements. To facilitate the understanding, we begin with these introductions of some requisite and foundational knowledge about multi-granulation interval-valued hesitant fuzzy rough set and information systems in Section 2. In Section 3, a new binary relation termed the average dominance relation is defined in the light of average dominance degree, whereafter an average dominance rough set is constructed in MG-IVHFIS and typical examples are illustrated to explain our model. We research four dynamic mechanisms caused by the alteration of attributes and design corresponding dynamic algorithms in Section 4. The Section 5 reveals all experimental results of dynamic and classical algorithms and evaluates the efficiency of both. Eventually, Section 6 ends up with a summary of full text.

2. Preliminaries

2.1. Interval-valued hesitant fuzzy set

In the light of hesitant fuzzy set and evaluations in the shape of interval value, interval-valued hesitant fuzzy set was primordially made an introduction by Chen and Cai [45], which was constructed by replacing single values in the hesitant fuzzy set with interval values. As a more flexible structure that reflects the hesitant degrees of experts when evaluating objects, the interval-valued hesitant fuzzy set deserves to be focused upon. What we will investigate next is hesitant fuzzy set, interval-valued hesitant fuzzy set and the approach of a comparison between interval-valued hesitant fuzzy elements.

Definition 2.1 (*See [53]*)**.** Let U be a universe with finite elements, then a hesitant fuzzy set (HFS) E over U is ruled as $E = \{\langle x, h_E(x)\rangle | x \in U\}$, where $h_E(x)$ is a set composed by several disparate and finite numbers in [0, 1], pointing out all possible membership degrees of x in U to set E, and the denotation of $h_E(x)$ is hesitant fuzzy element. $h_E(x)$ consisting of n ($n \geq 1$) numbers can be marked as $h_E(x) = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, where the number ε_i is in [0, 1].

After the presentation of Definition 2.1, we are about to review further information about interval-valued hesitant fuzzy set.

Definition 2.2 (*See [53]*)**.** Let U be a universe with finite elements, then an interval-valued hesitant fuzzy set (IVHFS) I over U is ruled as $I = \{\langle x, h_I(x)\rangle | x \in U\}$, where $h_I(x)$ is a set composed by several disparate and finite interval numbers in [0, 1], pointing out all possible interval-valued membership degrees of element x in U to set I, and the denotation of $h_I(x)$ is interval-valued hesitant fuzzy element. $h_I(x)$ consisting of n ($n \geq 1$) interval numbers can be marked as $h_I(x) = \{\nu_1, \nu_2, \ldots, \nu_n\}$, where the interval number is $\nu_i = [{\nu_i}^L, {\nu_i}^U]$. Here ${\nu_i}^L$, ${\nu_i}^U$ are the lower and upper limits of the interval.

With a view to the difference when people make an assessment of an incident, the quantity of interval numbers in diverse IVHF elements is perhaps diverse as well. Consequently, Xu and Da [54] provided a solution to estimate the magnitude between two interval numbers.

Definition 2.3 (*See [54]*)**.** Let two interval numbers be $p = [p^L, p^U]$ and $q = [q^L, q^U]$ respectively, and $p + q = [p^L + q^L, p^U + q^U]$, $l_p = p^U - p^L$, $l_q = q^U - q^L$, then the degree of possibility of $p \geq q$ and $p \leq q$ is denoted by

$$\begin{aligned} p(p \geq q) &= \max\{1 - \max(\frac{\mathrm{p}^U - q^L}{l_p + l_q}, 0), 0\}, \\ p(p \leq q) &= \max\{1 - \max(\frac{\mathrm{p}^U - q^L}{l_p + l_q}, 0), 0\}. \end{aligned} \tag{1}$$

If $p(p \geq q) > 0.5$, then p is superior to q, denoted by $p > q$. If $p(p \geq q) = 0.5$, then p is equivalent to q, denoted by $p = q$.

If given interval numbers are disordered, we may compare and sequence them employing the aforesaid definition. Whereupon, the following assumption is made [55]:

Let $h_U(x)$ and $h_V(x)$ be two IVHF elements. If $l(h_U(x)) \neq l(h_V(x))$, namely their lengths are inequable, then with the purpose of operating between them, the lengths of $h_U(x)$ and $h_V(x)$ should be equal as well as both of them are $l = \max\{l(h_U(x)), l(h_V(x))\}$. If $l(h_U(x)) < l(h_V(x))$, then $h_U(x)$ will be extended. In other words, we are requested to sequence its interval numbers and add its maximum interval value to $h_U(x)$ until $l(h_U(x)) = l(h_V(x))$.

2.2. Multi-granulation interval-valued hesitant fuzzy rough set

In accordance with the indiscernibility relation, these objects over U are clustered into basic knowledge utilizing RS, and roughness is characterized by upper and lower approximations. About this subsection, we will go over the essential concepts of RS firstly. Moreover, we mean to display the MGRS and extend the IVHFS to multi-granulation spaces.

Definition 2.4 (*See [1]*)**.** Let U be a universe with finite elements, and R is an equivalence relation over U. For any $X \subseteq U$, the lower and upper approximations of X with respect to R can be denoted by $\underline{R}(X)$ and $\overline{R}(X)$ respectively, which can be expressed as follows:

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\}, \qquad \overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\}. \tag{2}$$

They are abbreviated as the lower and upper approximations. If $\underline{R}(x) = \overline{R}(x)$, then X is a definable set, otherwise X is a rough set.

It is widely accepted that when a given universe is induced by multiple relations, these divisions formed can be regarded as multiple granularities, which determine the corresponding multi-granulation rough set (MGRS).

Definition 2.5 (*See [16]*)**.** Let $R_i (i = 1, 2, \ldots, s)$ be s equivalence relations over a universe U. For any $X \subseteq U$, the optimistic multi-granulation lower and upper approximations of X based on the relation $\sum_{i=1}^{s} R_i$ are defined as

$$\begin{aligned} \underline{apr}^O_{\sum_{i=1}^{s} R_i}(X) &= \{x \in U | \bigvee_{i=1}^{s} ([x]_{R_i} \subseteq X)\}, \\ \overline{apr}^O_{\sum_{i=1}^{s} R_i}(X) &= \{x \in U | \bigwedge_{i=1}^{s} ([x]_{R_i} \cap X \neq \emptyset)\}, \end{aligned} \tag{3}$$

where "$\vee$" signifies "*or*" and "$\wedge$" signifies "*and*". In addition, if $\underline{apr}^O_{\sum_{i=1}^{s} R_i}(X) = \overline{apr}^O_{\sum_{i=1}^{s} R_i}(X)$, then X is referred to be an optimistic multi-granulation definable set. Otherwise, X is an optimistic multi-granulation rough set.

Analogously, for any $X \subseteq U$, the pessimistic multi-granulation lower and upper approximations of X based on the relation $\sum_{i=1}^{s} R_i$ are defined as

$$\begin{aligned} \underline{apr}^P_{\sum_{i=1}^{s} R_i}(X) &= \{x \in U | \bigwedge_{i=1}^{s} ([x_i]_{R_i} \subseteq X)\}, \\ \overline{apr}^P_{\sum_{i=1}^{s} R_i}(X) &= \{x \in U | \bigvee_{i=1}^{s} ([x]_{R_i} \cap X \neq \emptyset)\}, \end{aligned} \tag{4}$$

if $\underline{apr}^P_{\sum_{i=1}^s R_i}(X) = \overline{apr}^P_{\sum_{i=1}^s R_i}(X)$, then X is referred to be a pessimistic multi-granulation definable set. Otherwise, X is a pessimistic multi-granulation rough set.

In combination with MGRS, we will further explore IVHFS under the environment of multiple granularities.

Definition 2.6 (*See [56]*)**.** Let $R_i\ (i = 1, 2, \ldots, s)$ be s IVHF relations over U. For any $A \in$ IVHFS, the optimistic multi-granulation interval-valued hesitant fuzzy lower and upper approximations of A based on the relation $\sum_{i=1}^s R_i$ are defined as

$$\begin{aligned}&\underline{M}^O_{\sum_{i=1}^s R_i}(A) = \{\langle x, h_{\underline{M}^O_{\sum_{i=1}^s R_i}(A)}(x)\rangle | x \in U\},\\ &\overline{M}^O_{\sum_{i=1}^s R_i}(A) = \{\langle x, h_{\overline{M}^O_{\sum_{i=1}^s R_i}(A)}(x)\rangle | x \in U\},\end{aligned} \tag{5}$$

where $h_{\underline{M}^O_{\sum_{i=1}^s R_i}(A)}(x) = \vee_{i=1}^s \wedge_{y\in U}\{h_{R_i^c}(x, y) \vee h_A(y)\}$, $h_{\overline{M}^O_{\sum_{i=1}^s R_i}(A)}(x) = \wedge_{i=1}^s \vee_{y\in U}\{h_{R_i}(x, y) \wedge h_A(y)\}$. "$\wedge$" signifies "*select smaller*" and "$\vee$" signifies "*select larger*". Additionally, if $\underline{M}^O_{\sum_{i=1}^s R_i}(A) = \overline{M}^O_{\sum_{i=1}^s R_i}(A)$, then A is optimistic and definable under multi-granulation relations. Otherwise, A is optimistic and rough.

Similarly, for any $A \in$ IVHFS, the pessimistic multi-granulation interval-valued hesitant fuzzy lower and upper approximations of A based on the relation $\sum_{i=1}^s R_i$ are defined as

$$\begin{aligned}&\underline{M}^P_{\sum_{i=1}^s R_i}(A) = \{\langle x, h_{\underline{M}^P_{\sum_{i=1}^s R_i}(A)}(x)\rangle | x \in U\},\\ &\overline{M}^P_{\sum_{i=1}^s R_i}(A) = \{\langle x, h_{\overline{M}^P_{\sum_{i=1}^s R_i}(A)}(x)\rangle | x \in U\},\end{aligned} \tag{6}$$

where $h_{\underline{M}^P_{\sum_{i=1}^s R_i}(A)}(x) = \wedge_{i=1}^s \wedge_{y\in U}\{h_{R_i^c}(x, y) \vee h_A(y)\}$, $h_{\overline{M}^P_{\sum_{i=1}^s R_i}(A)}(x) = \vee_{i=1}^s \vee_{y\in U}\{h_{R_i}(x, y) \wedge h_A(y)\}$. Additionally, if $\underline{M}^P_{\sum_{i=1}^s R_i}(A) = \overline{M}^P_{\sum_{i=1}^s R_i}(A)$, then A is pessimistic and definable under multi-granulation relations. Otherwise, A is pessimistic and rough.

2.3. Multi-granulation interval-valued hesitant fuzzy ordered information system

Generally speaking, classical RS is established on an information system, where is beneficial for researching practical issues. Accordingly, we will recall some relevant notions concerning information system initially.

Definition 2.7 (*See [1,2]*)**.** Let $I = (U, A, V, f)$ be an information system (IS). Here U is a universe with finite elements, and A is a non-empty set with n attributes $\{a_1, a_2, \ldots, a_n\}$. V is the domain of attributes marked as $V = \bigcup_{a\in A} V_a$, and f is an information function marked as $f : U \times A \to V$ where $f(x, a)$ is the value of object x under attribute a.

In an IS, an attribute is termed as a criterion when the domain of this attribute is completely ordered abiding by an increasing or a decreasing preference. Specifically, if all attributes are criterions, the IS is termed as an ordered information system (OIS). To make our statements more clear and explicit, only increasing preference is considered.

Definition 2.8 (*See [57]*)**.** Let $J = (U, A, V, f)$ be an IVHF information system (IVHFIS), then $f(x, a)$ is a IVHF value containing n interval numbers for any $a \in A$ and $x \in U$, marked by $f(x, a) = f(x_a) = \{[v_1^L, v_1^U], \ldots, [v_n^L, v_n^U]\}$ $(v_i^L, v_i^U \in R,\ i = 1, 2, \ldots, n)$.

In a given IVHFOIS, $f(x_a) = \{[v_1^L, v_1^U], \ldots, [v_n^L, v_n^U]\}$ and $f(y_a) = \{[w_1^L, w_1^U], \ldots, [w_n^L, w_n^U]\}$ $(n \geq 1)$. If $f(y_a) \geq f(x_a)$ for any $a \in A$, that shows y dominates x under a dominance relation $R_A^{\geq}$, which can be marked by $yR_A^{\geq}x$. Here, $R_A^{\geq} = \{(y, x) \in U \times U | f(x_a) \geq f(x_a), \forall a \in A\} = \{(y, x) \in U \times U | w_1^L \geq v_1^L, w_1^U \geq v_1^U, \ldots, w_n^L \geq v_n^L, w_n^U \geq v_n^U, \forall a \in A\}$ equipped with reflexivity, asymmetry and transitivity. We address that dominance class induced by dominance relation $R_A^{\geq}$ is constitutive of objects dominating x, which is defined as $[x]_A^{\geq} = \{y \in U | (x, y) \in R_A^{\geq}\}$.

In particular, for a given IVHFOIS, when $R_i (i = 1, 2, \ldots, s)$ is the ith dominance relation whose amount is s over U, the IVHFOIS is termed as a multi-granulation IVHFOIS.

Definition 2.9 (*See [58]*)**.** Let $K = (U, A, V, f)$ be a multi-granulation IVHFOIS, and $R_i (i = 1, 2, \ldots, s)$ is the ith dominance relation over U. For any $X \subseteq U$, the optimistic multi-granulation IVHF lower and upper approximations of X based on the relation $\sum_{i=1}^s R_i$ are defined as

$$\begin{aligned}&\underline{M}^O_{\sum_{i=1}^s R_i^{A\geq}}(X) = \{x \in U | \bigvee_{i=1}^{s}([x]_{R_i}^{A\geq} \subseteq X)\},\\ &\overline{M}^O_{\sum_{i=1}^s R_i^{A\geq}}(X) = \{x \in U | \bigwedge_{i=1}^{s}([x]_{R_i}^{A\geq} \cap X \neq \emptyset)\},\end{aligned} \tag{7}$$

where "$\vee$" signifies "*or*", "$\wedge$" signifies "*and*" and $[x]_{R_i}^{A\geq} = \{y \in U | (x, y) \in R_i^{A\geq}\}$. Here, $R_i^{A\geq}$ symbolizes a dominance relation under attribute A.

Likewise, for any $X \subseteq U$, the pessimistic multi-granulation IVHF lower and upper approximations of X based on the relation $\sum_{i=1}^s R_i$ are defined as

$$\begin{aligned}&\underline{M}^P_{\sum_{i=1}^s R_i^{A\geq}}(X) = \{x \in U | \bigwedge_{i=1}^{s}([x]_{R_i}^{A\geq} \subseteq X)\},\\ &\overline{M}^P_{\sum_{i=1}^s R_i^{A\geq}}(X) = \{x \in U | \bigvee_{i=1}^{s}([x]_{R_i}^{A\geq} \cap X \neq \emptyset)\}.\end{aligned} \tag{8}$$

In subsequent descriptions, some specific narratives in relation to the variation trend of dominance class under circumstance of multi-granulation when attributes change will be revealed.

Lemma 2.1 (*See [25]*)**.** *Let $K = (U, A, V, f)$ be a multi-granulation IVHFOIS, and $R_i(i = 1, 2, \ldots, s)$ is the ith dominance relation over U. For any $x \in U$ and attributes $l, m \in A$, it is observed that $[x]_{R_i}^{l\cup m\geq} \subseteq [x]_{R_i}^{l\geq}$, $[x]_{R_i}^{l\cup m\geq} \subseteq [x]_{R_i}^{m\geq}$ and $[x]_{R_i}^{l\cup m\geq} = [x]_{R_i}^{l\geq} \cap [x]_{R_i}^{m\geq}$.*

From Lemma 2.1, we may analyze and summarize that the more massive attributes are, the more refined the information is. Simultaneously, in a multi-granulation space, we ought to update timely the dominance class under each granularity as the subset of attributes transforms, namely certain attributes increase or decrease.

3. Average dominance rough set from IVHFIS to MG-IVHFIS

This section will center on a novel relation titled with average dominance relation, thereby the research of multi-granulation rough set established on this relation will be carry out.

3.1. Average dominance rough set in IVHFIS

Since the conditions of general dominance relation are extremely harse in an IVHFOIS and there are numerous restrictions in realistic application, the average dominance relation is brought forward. In the IVHF rough set, the average dominance degree between two objects is calculated to contrast whether one object is superior to the other.

Definition 3.1. In any IVHFIS $J = (U, A, V, F)$, for any $x_i, x_j \in U$ and $a \in A$, $f_a(x)$ represents the IVHF value of object x under attribute a. If $f_a(x_i) = \{[v_1^L, v_1^U], \ldots, [v_n^L, v_n^U]\}$, $f_a(x_j) = \{[w_1^L, w_1^U], \ldots, [w_m^L, w_m^U]\}$ $(n,\ m \geq 1)$, then the average dominance degree between x_i and x_j under attribute a is ruled as given in Eq. (9)

$$D_{\mathrm{a}}^{\geq}(x_i,x_j)=\frac{1}{N}\sum_{k=1}^{N}\min\{1,\max\{\frac{\frac{1}{2}[\min(v_k^U,w_k^U)-\max(v_k^L,w_k^L)]+\max(v_k^U-w_k^U,0)+\max(v_k^L-w_k^L,0)}{\max(v_k^U,w_k^U)-\min(v_k^L,w_k^L)},0\}\}, \tag{9}$$

Box I.

Table 1
An average dominance relation $R_1^{A\geq}$ in MG-IVHFIS.

$R_1^{A\geq}$	a_1	a_2	a_3	a_4	a_5
x_1	[80, 83], [81, 84]	[85, 88], [87, 89]	[83, 86], [85, 87]	[87, 88], [89, 90]	[71, 75], [74, 77]
x_2	[76, 84], [78, 85]	[83, 84], [82, 87]	[85, 87], [86, 89]	[87, 90], [88, 91]	[78, 82], [77, 83]
x_3	[89, 92], [90, 94]	[82, 85], [84, 86]	[92, 95], [93, 96]	[93, 95], [92, 97]	[85, 88], [87, 92]
x_4	[85, 87], [86, 89]	[90, 93], [91, 96]	[87, 90], [88, 91]	[91, 92], [93, 95]	[87, 89], [88, 93]
x_5	[72, 74], [73, 76]	[89, 94], [92, 95]	[88, 89], [89, 90]	[86, 89], [87, 91]	[77, 83], [78, 84]

given in Box I, where n and m are the quantity of interval numbers in $f_a(x_i)$ and $f_a(x_j)$ namely $n = l(f_a(x_i))$, $m = l(f_a(x_j))$, and $N = \max[l(f_a(x_i)), l(f_a(x_j))]$. It should be noted that when $l(f_a(x_i)) \neq l(f_a(x_j))$, we are demanded to extend the IVHF value until $l(f_a(x_i)) = l(f_a(x_j))$ according to Definition 2.3.

Proposition 3.1. *For the average dominance degree $D_{\mathrm{a}}^{\geq}(x_i,x_j)$, the following properties are satisfied:*

(1) $0 \leq D_{\mathrm{a}}^{\geq}(x_i,x_j) \leq 1$.
(2) $D_{\mathrm{a}}^{\geq}(x_i,x_j) + D_{\mathrm{a}}^{\geq}(x_i,x_j) = 1$.
(3) $D_{\mathrm{a}}^{\geq}(x_i,x_j) = 0.5 \Leftrightarrow f_a(x_i) = f_a(x_j)$.

Distinctly, if object x_i is superior to x_j, $D_{\mathrm{a}}^{\geq}(x_i,x_j) \geq 0.5$. If object x_i is inferior to x_j, $D_{\mathrm{a}}^{\geq}(x_i,x_j) < 0.5$. Further, the definition of average dominance degree between x_i and x_j with respect to the attribute set A is $D_A^{\geq}(x_i,x_j) = \min_{a\in A}\{D_{\mathrm{a}}^{\geq}(x_i,x_j)\}$.

Definition 3.2. Given an IVHFIS = (U, A, V, f) with $U = \{x_1, x_2, \ldots, x_n\}$, for any $B \subseteq A$, the average dominance relation $R_B^{\widehat{\geq}}$ is ruled as

$$R_B^{\widehat{\geq}} = \{(x_i,x_j) \in U \times U | D_B^{\geq}(x_i,x_j) \geq 0.5\}, \tag{10}$$

and the average dominance class of x_i on the basis of this relation is ruled as

$$[x_i]_B^{\widehat{\geq}} = \{x_j \in U | (x_i,x_j) \in R_B^{\widehat{\geq}}\}, \tag{11}$$

which comprises all objects dominating x_i according to average dominance relation. It apparently comes to a conclusion that the average dominance relation is reflexive, asymmetrical and transitive, and also a cover over U is constituted by all of the average dominance classes.

Definition 3.3. Given an IVHFIS = (U, A, V, f) with $U = \{x_1, x_2, \ldots, x_n\}$, for any $B \subseteq A$ and $X \subseteq U$, the IVHF lower and upper approximations of X based on the relation $R_B^{\widehat{\geq}}$ are defined as

$$\underline{R}_B^{\widehat{\geq}}(X) = \{x \in U | [x]_B^{\widehat{\geq}} \subseteq X\}, \qquad \overline{R}_B^{\widehat{\geq}}(X) = \{x \in U | [x]_B^{\widehat{\geq}} \cap X \neq \emptyset\}. \tag{12}$$

In line with classical RS, if $\underline{R}_B^{\widehat{\geq}}(X) \neq \overline{R}_B^{\widehat{\geq}}(X)$, X is termed as an average dominance rough set in IVHFIS. Otherwise, X is definable under average dominance relation.

3.2. Average dominance rough set in MG-IVHFIS

We introduce the average dominance rough set into IVHFIS in the previous section. What we aim to study next is the average dominance under multi-granulation environment. As an extension of IVHFIS, the multi-granulation IVHFIS (MG-IVHFIS) involves more requirements of people in diverse granularity spaces.

Definition 3.4. Given a MG-IVHFIS = (U, A, V, f) with $U = \{x_1, x_2, \ldots, x_n\}$, $R_i (i = 1, 2, \ldots, s)$ is the ith average dominance relation over U. For any $X \subseteq U$, the optimistic MG-IVHF lower and upper approximations of X based on the average dominance relation $\sum_{i=1}^{s} R_i$ are defined as

$$\begin{aligned}
&\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) = \{x \in U | \bigvee_{i=1}^{s}([x]_{R_i}^{A\widehat{\geq}} \subseteq X)\},\\
&\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) = \{x \in U | \bigwedge_{i=1}^{s}([x]_{R_i}^{A\widehat{\geq}} \cap X \neq \emptyset)\},
\end{aligned} \tag{13}$$

where "$\vee$" signifies "*or*", "$\wedge$" signifies "*and*" and $[x]_{R_i}^{A\widehat{\geq}} = \{y \in U | (x,y) \in R_i^{A\widehat{\geq}}\}$. Here, $R_i^{A\widehat{\geq}}$ symbolizes an average dominance relation under attribute A.

Analogously, for any $X \subseteq U$, the pessimistic MG-IVHF lower and upper approximations of X based on the average dominance relation $\sum_{i=1}^{s} R_i$ are defined as

$$\begin{aligned}
&\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) = \{x \in U | \bigwedge_{i=1}^{s}([x]_{R_i}^{A\widehat{\geq}} \subseteq X)\},\\
&\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) = \{x \in U | \bigvee_{i=1}^{s}([x]_{R_i}^{A\widehat{\geq}} \cap X \neq \emptyset)\}.
\end{aligned} \tag{14}$$

Equally, we address X an optimistic average dominance rough set in MG-IVHFIS when $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \neq \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$ and X a pessimistic average dominance rough set in MG-IVHFIS when $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \neq \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$.

In what follows, we demonstrate an instance to explore the average dominance rough set in MG-IVHFIS.

Example 3.1. As illustrated in Table 1, Table 2 and Table 3, it is a complete MG-IVHFIS comprising three average dominance relation. Connected with automotive evaluation, this case reveals logical scores provided by professionals on a centesimal system. Here, $U = \{x_1, x_2, x_3, x_4, x_5\}$ representative of five hot-selling automobiles in disparate models from the same motor corporation. $A = \{a_1, a_2, a_3, a_4, a_5\}$ representative of five evaluation indicators, where a_i $(i = 1, 2, 3, 4, 5)$ represent appearance, comfort level, power performance, brake performance and fuel economy. $R_j^{A\widehat{\geq}}$ $(j = 1, 2, 3)$ stand for respective scores from three professionals.

In the beginning, we are able to acquire all the average dominance classes by computation:

Table 2
An average dominance relation $R_2^{A\widehat{\succeq}}$ in MG-IVHFIS.

$R_2^{A\widehat{\succeq}}$	a_1	a_2	a_3	a_4	a_5
x_1	[78, 81], [80, 83]	[81, 83], [85, 86]	[83, 84], [86, 87]	[85, 88], [86, 90]	[67, 70], [68, 72]
x_2	[76, 82], [77, 83]	[78, 80], [81, 82]	[91, 94], [92, 95]	[87, 90], [89, 91]	[73, 76], [76, 78]
x_3	[80, 85], [85, 87]	[90, 92], [94, 96]	[93, 96], [94, 97]	[94, 96], [95, 96]	[81, 82], [82, 84]
x_4	[81, 84], [83, 87]	[92, 93], [91, 94]	[82, 85], [84, 89]	[86, 87], [87, 91]	[66, 70], [70, 71]
x_5	[75, 76], [74, 77]	[86, 89], [88, 90]	[84, 88], [86, 89]	[83, 85], [84, 87]	[77, 78], [78, 80]

Table 3
An average dominance relation $R_3^{A\widehat{\succeq}}$ in MG-IVHFIS.

$R_3^{A\widehat{\succeq}}$	a_1	a_2	a_3	a_4	a_5
x_1	[81, 83], [82, 84]	[86, 89], [90, 91]	[81, 83], [85, 87]	[92, 94], [93, 94]	[79, 80], [80, 83]
x_2	[77, 82], [79, 81]	[84, 85], [86, 87]	[84, 85], [86, 87]	[89, 90], [90, 92]	[72, 74], [73, 76]
x_3	[90, 92], [91, 94]	[89, 91], [93, 95]	[89, 91], [90, 93]	[92, 95], [94, 97]	[87, 89], [88, 92]
x_4	[86, 87], [87, 88]	[90, 93], [91, 92]	[86, 88], [88, 92]	[88, 91], [90, 93]	[73, 75], [74, 76]
x_5	[73, 75], [74, 76]	[87, 88], [89, 91]	[82, 83], [84, 86]	[88, 91], [89, 92]	[78, 81], [81, 82]

$[x_1]_{R_1}^{A\widehat{\succeq}} = \{x_1, x_4\}$, $[x_2]_{R_1}^{A\widehat{\succeq}} = \{x_2, x_3, x_4\}$, $[x_3]_{R_1}^{A\widehat{\succeq}} = \{x_3\}$, $[x_4]_{R_1}^{A\widehat{\succeq}} = \{x_4\}$, $[x_5]_{R_1}^{A\widehat{\succeq}} = \{x_4, x_5\}$.

$[x_1]_{R_2}^{A\widehat{\succeq}} = \{x_1, x_3, x_4\}$, $[x_2]_{R_2}^{A\widehat{\succeq}} = \{x_2, x_3\}$, $[x_3]_{R_2}^{A\widehat{\succeq}} = \{x_3\}$, $[x_4]_{R_2}^{A\widehat{\succeq}} = \{x_3, x_4\}$, $[x_5]_{R_2}^{A\widehat{\succeq}} = \{x_3, x_5\}$.

$[x_1]_{R_3}^{A\widehat{\succeq}} = \{x_1, x_3\}$, $[x_2]_{R_3}^{A\widehat{\succeq}} = \{x_2, x_3, x_4\}$, $[x_3]_{R_3}^{A\widehat{\succeq}} = \{x_3\}$, $[x_4]_{R_3}^{A\widehat{\succeq}} = \{x_3, x_4\}$, $[x_5]_{R_3}^{A\widehat{\succeq}} = \{x_1, x_3, x_5\}$.

Provided that a set $X = \{x_1, x_3, x_5\}$, we may figure the lower and upper approximations of X based on the average dominance relation in MG-IVHFIS sequentially.

Conforming to aforementioned definitions, we calculate readily the optimistic lower and upper approximations:

$\underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\succeq}}}(X) = \{x_1, x_3, x_5\}$, $\overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\succeq}}}(X) = \{x_1, x_2, x_3, x_5\}$.

In a similar means, the calculations in the pessimistic situation are as follows:

$\underline{M}^P_{\sum_{i=1}^s R_i^{A\widehat{\succeq}}}(X) = \{x_3\}$, $\overline{M}^P_{\sum_{i=1}^s R_i^{A\widehat{\succeq}}}(X) = \{x_1, x_2, x_3, x_4, x_5\}$.

4. The mechanism for updating approximations in dynamic MG-IVHFIS

Intricate information is ever-changing in this era of big data. Confronted with evolving data over time, it should be noted that adding the latest information and deleting the redundant information timely. As a crucial strategy in the field of data mining, incremental technique is exerted to dispose of data updating with effect. In MG-IVHFIS, the attribute set possibly vary over time, which includes increase or decrease. Relying on the incremental technique, it is not necessary for us to recalculate the upper and lower approximations, which is an excellent time-saving measure. This chapter considers dynamic updating approximations method by inserting or removing some attributes while these objects maintain invariable in MG-IVHFIS.

4.1. The variation trend of approximations as attributes vary

What matters is that a variation in the attribute set results in a variation in average dominance classes, which leads to a variation in approximations. In consideration of this, we will raise several propositions to reveal the variation trend of approximations with attributes changing and further reflect the variation intuitively with examples.

Proposition 4.1. *Given a MG-IVHFIS = (U, A, V, f), $R_i (i = 1, 2, \ldots, s)$ is the ith average dominance relation over U. For any $X \subseteq U$, $C \subseteq B$ and $B, C \subseteq A$, the following properties are satisfied:*

(OL) $\underline{M}^O_{\sum_{i=1}^s R_i^{B-C\widehat{\succeq}}}(X) \subseteq \underline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X)$,

(OU) $\overline{M}^O_{\sum_{i=1}^s R_i^{B-C\widehat{\succeq}}}(X) \supseteq \overline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X)$.

(PL) $\underline{M}^P_{\sum_{i=1}^s R_i^{B-C\widehat{\succeq}}}(X) \subseteq \underline{M}^P_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X)$,

(PU) $\overline{M}^P_{\sum_{i=1}^s R_i^{B-C\widehat{\succeq}}}(X) \supseteq \overline{M}^P_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X)$.

Proof. Aimed at facilitating the description, the proof is only presented under circumstance of two granulations. Assume two average dominance relations are $R_i^{\widehat{\succeq}}$, $R_j^{\widehat{\succeq}}$ respectively in MG-IVHFIS. When the attribute subset C is deleted from B:

(OL) Knowing from Definition 3.3, for every $x \in \underline{R_i}^{B-C\widehat{\succeq}}(X)$, it is distinct that $[x]_{R_i}^{B-C\widehat{\succeq}} \subseteq X$. Similarly, we still have $[x]_{R_i}^{B\widehat{\succeq}} \subseteq [x]_{R_i}^{B-C\widehat{\succeq}}$. Thereupon, $[x]_{R_i}^{B\widehat{\succeq}} \subseteq X$, which implies $x \in \underline{R_i}^{B\widehat{\succeq}}(X)$, then $\underline{R_i}^{B-C\widehat{\succeq}}(X) \subseteq \underline{R_i}^{B\widehat{\succeq}}(X)$. Consequently, $\underline{R_i}^{B-C\widehat{\succeq}}(X) \vee \underline{R_j}^{B-C\widehat{\succeq}}(X) \subseteq \underline{R_i}^{B\widehat{\succeq}}(X) \vee \underline{R_j}^{B\widehat{\succeq}}(X)$. Namely, $\underline{M}^O_{\sum_{i=1}^s R_i^{B-C\widehat{\succeq}}}(X) \subseteq \underline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X)$.

(OU) As is derived from Definition 3.3, for every $x \in \overline{R_i}^{B\widehat{\succeq}}(X)$, we can infer that $[x]_{R_i}^{B\widehat{\succeq}} \cap X \neq \emptyset$. Coupled with Lemma 2.1, it is easily obtainable that $[x]_{R_i}^{B\widehat{\succeq}} \subseteq [x]_{R_i}^{B-C\widehat{\succeq}}$. So $[x]_{R_i}^{B-C\widehat{\succeq}} \cap X \neq \emptyset$, which implies $x \in \overline{R_i}^{B-C\widehat{\succeq}}(X)$, then $\overline{R_i}^{B-C\widehat{\succeq}}(X) \supseteq \overline{R_i}^{B\widehat{\succeq}}(X)$. Accordingly, $\overline{R_i}^{B-C\widehat{\succeq}}(X) \wedge \overline{R_j}^{B-C\widehat{\succeq}}(X) \supseteq \overline{R_i}^{B\widehat{\succeq}}(X) \wedge \overline{R_j}^{B\widehat{\succeq}}(X)$. That is to say, $\overline{M}^O_{\sum_{i=1}^s R_i^{B-C\widehat{\succeq}}}(X) \supseteq \overline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X)$.

Without loss of generality, we merely demonstrate properties under optimistic situation, then it is uncomplicated to obtain (PL) and (PU).

Proposition 4.2. *Given a MG-IVHFIS (U, A, V, f), $R_i (i = 1, 2, \ldots, s)$ is the ith average dominance relation over U. For any $X \subseteq U$ and $B, C \subseteq A$, the following properties are satisfied:*

(OL) $\underline{M}^O_{\sum_{i=1}^s R_i^{B\cup C\widehat{\succeq}}}(X) \supseteq \underline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X) \cup \underline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\succeq}}}(X)$,

(OU) $\overline{M}^O_{\sum_{i=1}^s R_i^{B\cup C\widehat{\succeq}}}(X) \subseteq \overline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X) \cap \overline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\succeq}}}(X)$.

(PL) $\underline{M}^P_{\sum_{i=1}^s R_i^{B\cup C\widehat{\succeq}}}(X) \supseteq \underline{M}^P_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X) \cup \underline{M}^P_{\sum_{i=1}^s R_i^{C\widehat{\succeq}}}(X)$,

(PU) $\overline{M}^P_{\sum_{i=1}^s R_i^{B\cup C\widehat{\succeq}}}(X) \subseteq \overline{M}^P_{\sum_{i=1}^s R_i^{B\widehat{\succeq}}}(X) \cap \overline{M}^P_{\sum_{i=1}^s R_i^{C\widehat{\succeq}}}(X)$.

Proof. When the attribute subset C is added to B:

(OL) According to Definition 3.3, for every $x \in \underline{R}_i^{B\widehat{\geq}}(X) \vee \underline{R}_i^{C\widehat{\geq}}(X)$, it is equivalent to $x \in \underline{R}_i^{B\widehat{\geq}}(X)$ and $x \in \underline{R}_i^{C\widehat{\geq}}(X)$. Hence, $[x]_{R_i}^{B\widehat{\geq}} \subseteq X$ and $[x]_{R_i}^{C\widehat{\geq}} \subseteq X$. We may get that $[x]_{R_i}^{B\cup C\widehat{\geq}} \subseteq [x]_{R_i}^{B\widehat{\geq}}$ and $[x]_{R_i}^{B\cup C\widehat{\geq}} \subseteq [x]_{R_i}^{C\widehat{\geq}}$ associated with Lemma 2.1. So $[x]_{R_i}^{B\cup C\widehat{\geq}} \subseteq X$, that is $x \in \underline{R}_i^{B\cup C\widehat{\geq}}(X)$. Thus, $\underline{R}_i^{B\widehat{\geq}}(X) \vee \underline{R}_i^{C\widehat{\geq}} \subseteq \underline{R}_i^{B\cup C\widehat{\geq}}$ can be proved. Consequently, $[\underline{R}_i^{B\widehat{\geq}}(X) \vee \underline{R}_j^{B\widehat{\geq}}] \vee [\underline{R}_i^{C\widehat{\geq}}(X) \vee \underline{R}_j^{C\widehat{\geq}}] \subseteq \underline{R}_i^{B\cup C\widehat{\geq}} \vee \underline{R}_j^{B\cup C\widehat{\geq}}$. Finally, $\underline{M}^O_{\sum_{i=1}^s R_i^{B\cup C\widehat{\geq}}}(X) \supseteq \underline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) \cup \underline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X)$ is achieved.

(OU) As is known from Definition 3.3, for every $x \in \overline{R}_i^{B\cup C\widehat{\geq}}(X)$, we have $[x]_{R_i}^{B\cup C\widehat{\geq}} \cap X \neq \emptyset$. Coupled with Lemma2.1, $[x]_{R_i}^{B\cup C\widehat{\geq}} \subseteq [x]_{R_i}^{B\widehat{\geq}}$ and $[x]_{R_i}^{B\cup C\widehat{\geq}} \subseteq [x]_{R_i}^{C\widehat{\geq}}$. It is apparent that $[x]_{R_i}^{B\widehat{\geq}} \cap X \neq \emptyset$ and $[x]_{R_i}^{C\widehat{\geq}} \cap X \neq \emptyset$. That indicates $x \in \overline{R}_i^{B\widehat{\geq}}(X)$ and $x \in \overline{R}_i^{C\widehat{\geq}}(X)$, namely $x \in \overline{R}_i^{B\widehat{\geq}}(X) \wedge \overline{R}_i^{C\widehat{\geq}}(X)$. So $\overline{R}_i^{B\cup C\widehat{\geq}}(X) \subseteq \overline{R}_i^{B\widehat{\geq}}(X) \wedge \overline{R}_i^{C\widehat{\geq}}(X)$. Accordingly, $\overline{R}_i^{B\cup C\widehat{\geq}}(X) \wedge \overline{R}_j^{B\cup C\widehat{\geq}}(X) \subseteq [\overline{R}_i^{B\widehat{\geq}}(X) \wedge \overline{R}_j^{B\widehat{\geq}}(X)] \wedge [\overline{R}_i^{C\widehat{\geq}}(X) \wedge \overline{R}_j^{C\widehat{\geq}}(X)]$. Thereby, $\overline{M}^O_{\sum_{i=1}^s R_i^{B\cup C\widehat{\geq}}}(X) \subseteq \overline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) \cap \overline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X)$ is proved.

The rest of propositions, (PL) and (PU), can be demonstrated in a similar pattern.

From above-mentioned propositions, we summarize that if some attributes are deleted from the attribute set, the upper approximation tends to become larger while the lower approximation tends to become smaller. On the contrary, if some attributes are added to the attribute set, the upper approximation tends to become smaller while the lower approximation tends to become larger. With the purpose of making this conclusion more legible, we continue to expound it by an example.

Example 4.1 (*Continued from Example 3.1*)**.** Firstly, we select two attribute subsets B and C, where $B = \{a_2, a_4, a_5\}$ and $C = \{a_1, a_3\}$. Next, the average dominance classes are computed. The calculation results under attribute subset B are shown:

$[x_1]_{R_1}^{B\widehat{\geq}} = \{x_1, x_4\}$, $[x_2]_{R_1}^{B\widehat{\geq}} = \{x_2, x_3, x_4\}$, $[x_3]_{R_1}^{B\widehat{\geq}} = \{x_3\}$, $[x_4]_{R_1}^{B\widehat{\geq}} = \{x_4\}$, $[x_5]_{R_1}^{B\widehat{\geq}} = \{x_4, x_5\}$.

$[x_1]_{R_2}^{B\widehat{\geq}} = \{x_1, x_3, x_4\}$, $[x_2]_{R_2}^{B\widehat{\geq}} = \{x_2, x_3\}$, $[x_3]_{R_2}^{B\widehat{\geq}} = \{x_3\}$, $[x_4]_{R_2}^{B\widehat{\geq}} = \{x_3, x_4\}$, $[x_5]_{R_2}^{B\widehat{\geq}} = \{x_3, x_5\}$.

$[x_1]_{R_3}^{B\widehat{\geq}} = \{x_1, x_3\}$, $[x_2]_{R_3}^{B\widehat{\geq}} = \{x_1, x_2, x_3, x_4\}$, $[x_3]_{R_3}^{B\widehat{\geq}} = \{x_3\}$, $[x_4]_{R_3}^{B\widehat{\geq}} = \{x_3, x_4\}$, $[x_5]_{R_3}^{B\widehat{\geq}} = \{x_1, x_3, x_5\}$.

Then, it is toilless to get the lower and upper approximations concerning X under B:

$\underline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) = \{x_1, x_3, x_5\}$, $\overline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) = \{x_1, x_2, x_3, x_5\}$, $\underline{M}^P_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) = \{x_3\}$, $\overline{M}^P_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) = \{x_1, x_2, x_3, x_4, x_5\}$.

Furthermore, we can acquire the calculation results under attribute subset C equally:

$[x_1]_{R_1}^{C\widehat{\geq}} = \{x_1, x_3, x_4\}$, $[x_2]_{R_1}^{C\widehat{\geq}} = \{x_2, x_3, x_4\}$, $[x_3]_{R_1}^{C\widehat{\geq}} = \{x_3\}$, $[x_4]_{R_1}^{C\widehat{\geq}} = \{x_3, x_4\}$, $[x_5]_{R_1}^{C\widehat{\geq}} = \{x_3, x_4, x_5\}$.

$[x_1]_{R_2}^{C\widehat{\geq}} = \{x_1, x_3, x_4\}$, $[x_2]_{R_2}^{C\widehat{\geq}} = \{x_2, x_3\}$, $[x_3]_{R_2}^{C\widehat{\geq}} = \{x_3\}$, $[x_4]_{R_2}^{C\widehat{\geq}} = \{x_3, x_4\}$, $[x_5]_{R_2}^{C\widehat{\geq}} = \{x_2, x_3, x_5\}$.

$[x_1]_{R_3}^{C\widehat{\geq}} = \{x_1, x_3, x_4\}$, $[x_2]_{R_3}^{C\widehat{\geq}} = \{x_2, x_3, x_4\}$, $[x_3]_{R_3}^{C\widehat{\geq}} = \{x_3\}$, $[x_4]_{R_3}^{C\widehat{\geq}} = \{x_3, x_4\}$, $[x_5]_{R_3}^{C\widehat{\geq}} = \{x_1, x_2, x_3, x_4, x_5\}$.

By computation, the lower and upper approximations concerning X under C is following:

$\underline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X) = \{x_3\}$, $\overline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X) = \{x_1, x_2, x_3, x_4, x_5\}$, $\underline{M}^P_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X) = \{x_3\}$, $\overline{M}^P_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X) = \{x_1, x_2, x_3, x_4, x_5\}$.

Thus, we can observe from these results that $\underline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) \subseteq \underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$, $\overline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) \supseteq \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$, $\underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \supseteq \underline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) \cup \underline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X)$, $\overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \subseteq \overline{M}^O_{\sum_{i=1}^s R_i^{B\widehat{\geq}}}(X) \cap \overline{M}^O_{\sum_{i=1}^s R_i^{C\widehat{\geq}}}(X)$ and so on, which are consistent with Propositions 4.1 and 4.2.

Updating approximations by employing a static method is complex and inefficient. In order to save valuable time, we will study two mechanisms of dynamic updating approximations result from the change of attributes. During the process of updating, what we request to consider is two aspects, namely the original MG-IVHFIS and the changed MG-IVHFIS. Here, an original MG-IVHFIS is marked by $I^{\widehat{\geq}} = (U, A, V, f)$, and a changed MG-IVHFIS is marked by $I'^{\widehat{\geq}} = (U, A', V, f)$.

4.2. The updating mechanism about deleting some attributes

The subsequent research has to do with the incremental mechanism when some attributes are removed from original MG-IVHFIS. The attribute set composed of deleted attributes is denoted by B. Moreover, "$\top$" and "$\bot$" express the variation of upper and lower approximations.

Proposition 4.3. *Let $I^{\widehat{\geq}} = (U, A, V, f)$ be a MG-IVHFIS. For any $X \subseteq U$ and $B \subseteq A$, we can update the lower and upper approximations of X as follows:*

(1) $\underline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) = \underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) - \bot$, *where* $\bot = \{x \in \underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \mid \wedge_{i=1}^s([x]_{R_i}^{A-B\widehat{\geq}} \not\subseteq X)\}$.

(2) $\overline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) = \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \cup \top$, *where* $\top = \{x \in (U - \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)) \mid \wedge_{i=1}^s([x]_{R_i}^{A-B\widehat{\geq}} \cap X \neq \emptyset)\}$.

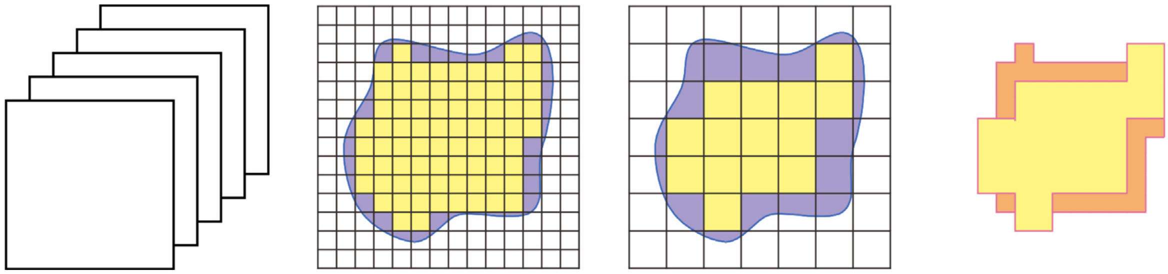
(3) $\underline{M}^P_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) = \underline{M}^P_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) - \bot$, *where* $\bot = \{x \in \underline{M}^P_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \mid \vee_{i=1}^s([x]_{R_i}^{A-B\widehat{\geq}} \not\subseteq X)\}$.

(4) $\overline{M}^P_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) = \overline{M}^P_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \cup \top$, *where* $\top = \{x \in (U - \overline{M}^P_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)) \mid \vee_{i=1}^s([x]_{R_i}^{A-B\widehat{\geq}} \cap X \neq \emptyset)\}$.

Proof. Without loss of generality, we merely demonstrate properties under optimistic situation.

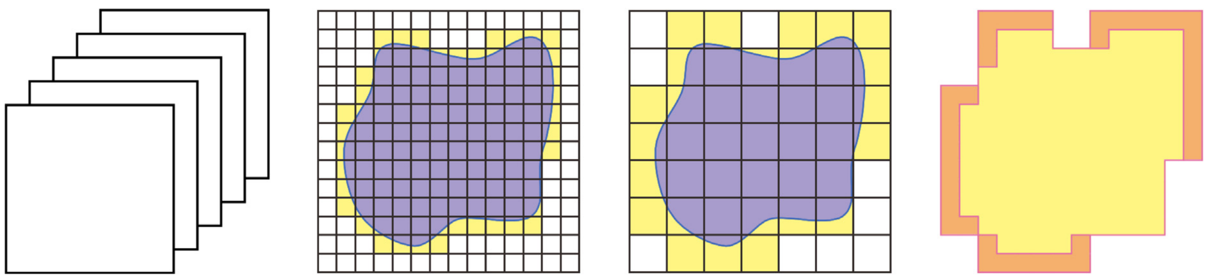
(1) Since it is evident that $\underline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) \subseteq \underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$ depending on Proposition 4.1, there is a set $\bot$ making that $\underline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) = \underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) - \bot$. That indicates we are going to examine objects, which are in $\underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$ before removing attribute set B. Besides, for s granularity spaces, object x from $\underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$ is deleted only if $[x]_{R_i}^{A-B\widehat{\geq}} \not\subseteq X$ in each granularity space. That is to say, $\bot = \{x \in \underline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \mid \wedge_{i=1}^s([x]_{R_i}^{A-B\widehat{\geq}} \not\subseteq X)\}$.

(2) What we know is that $\overline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) \supseteq \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$ on the basis of Proposition 4.1. So it is similar that a set $\top$ plays



(a) The multi-granulation space in MG-IVHFIS (b) The original lower approximation about $R_i^{A\widehat{\geq}}$ (c) The updated lower approximation after a deletion (d) The specific variation

Fig. 2. The process of updating the lower approximation when some attributes are deleted in MG-IVHFIS.



(a) The multi-granulation space in MG-IVHFIS (b) The original upper approximation about $R_i^{A\widehat{\geq}}$ (c) The updated upper approximation after a deletion (d) The specific variation

Fig. 3. The process of updating the upper approximation when some attributes are deleted in MG-IVHFIS.

a crucial role so that $\overline{M}^O_{\sum_{i=1}^s R_i^{A-B\widehat{\geq}}}(X) = \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X) \cup \top$. This suggests objects which do not belong to $\overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$ before deleting attribute set B are considered, namely objects belonging to $U - \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)$ should be examined. Then, for every granularity space, if there is an object x fulfilling that $[x]_{R_i}^{A-B\widehat{\geq}} \cap X \neq \emptyset$ after the deletion, we will insert object x into the upper approximation. That is to say, $\top = \{x \in (U - \overline{M}^O_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)) | \wedge_{i=1}^s([x]_{R_i}^{A-B\widehat{\geq}} \cap X \neq \emptyset)\}$.

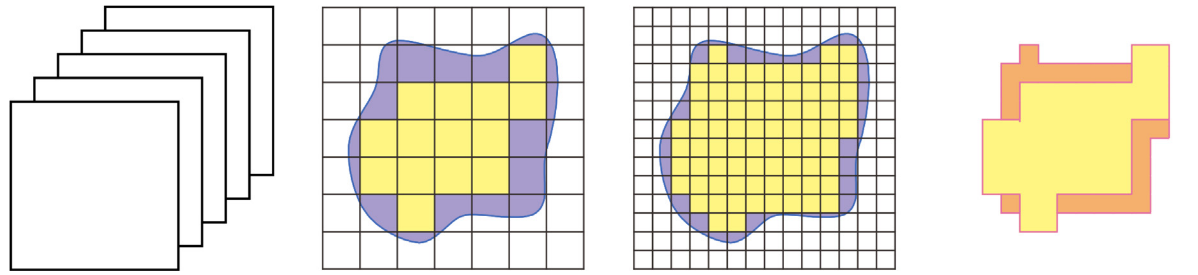
In order to present the updating mechanism more clearly and intuitively, we make use of Figs. 2 and 3 to depict concrete processes. What should be paid attention to is that we exert a partition instead of a covering over U designed to facilitate the description. As illustrated in these figures, Figs. 2 and 3 show the process of updating approximations when some attributes are deleted in MG-IVHFIS. In Fig. 2, we can observe the variation of lower approximation, among which (a) stands for the multi-granulation space, (b) and (c) reveal the original and updated lower approximations. The orange part of (d) represents these objects removed from the original lower approximation, namely the set $\perp$. It is noticeable that the updated lower approximation becomes smaller after a deletion. Consistent with Fig. 2, Fig. 3 displays the variation of upper approximation. Here, (a) represents the multi-granulation space, (b) and (c) reveal the upper approximation before and after deleting attribute set B concerning the ith average dominance relation in MG-IVHFIS. About (d), we draw these objects added to the original lower approximation in orange, namely the set $\top$. That means the updated upper approximation becomes larger after a deletion.

By summarizing above-mentioned discussions, a relevant algorithm is provided to update approximations when deleting some attributes. In Algorithm 1, we introduce the mechanism for dynamic updating under situation of optimistic multi-granulation.

Table 4
The time complexity of Algorithm 1 and Algorithm 2.

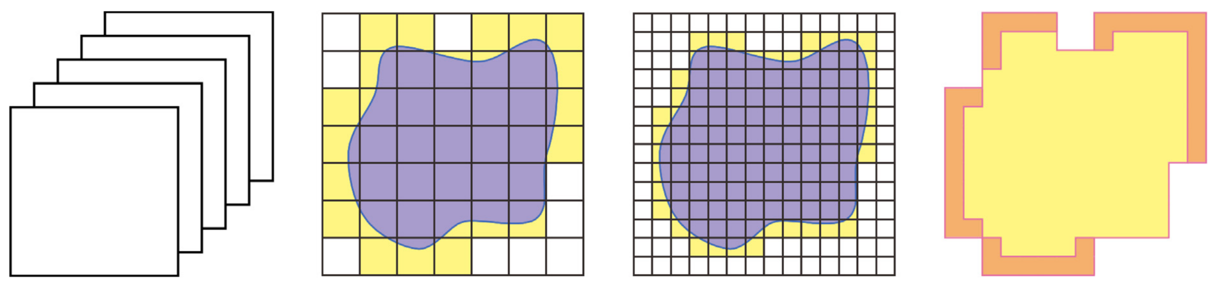
Steps	*Time complexity*
2–5	$O(\lvert s\rvert\lvert U\rvert^2)$
6–18	$O(\lvert s\rvert\lvert \underline{M}^{O(P)}_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)\rvert)$
19	$O(1)$
20–32	$O(\lvert s\rvert\lvert U - \overline{M}^{O(P)}_{\sum_{i=1}^s R_i^{A\widehat{\geq}}}(X)\rvert)$
33	$O(1)$
Total	$O(\lvert s\rvert\lvert U\rvert^2)$

Step 2 is the initialization of $\perp$ and $\top$. In steps 3–5, we calculate new average dominance classes for all objects over U in s granularity spaces. Steps 6–19 update the optimistic lower approximation of X, among which steps 8–13 make a judgment whether $[x]_{R_i}^{A-B\widehat{\geq}} \not\subseteq X$ for each granularity, and "mark" is assigned to two values (0 or 1) in disparate situations. If there is a granularity satisfying that $[x]_{R_i}^{A-B\widehat{\geq}} \subseteq X$, This indicates that object x cannot be removed. Therefore, "mark" is assigned to 0 and the loop is broken out. In steps 20–33, the optimistic upper approximation of X is updated, among which steps 22–27 judge whether there is an arbitrary granularity satisfying that $[x]_{R_i}^{A-B\widehat{\geq}} \cap X = \emptyset$. If so, object x will not be added to new upper approximation. Accordingly, "mark" is assigned to 0 and the loop is broken out. Finally, new optimistic MG-IVHF lower and upper approximations are output. By imitating Algorithm 1, it is not hard to discover that the incremental algorithm of pessimistic multi-granulation is similar, so Algorithm 2 may be viewed for more details. The time complexity of Algorithm 1 and Algorithm 2 are shown in Table 4.



(a) The multi-granulation space in MG-IVHFIS (b) The original lower approximation about $R_i^{A\widehat{\geq}}$ (c) The updated lower approximation after an addition (d) The specific variation

Fig. 4. The process of updating the lower approximation when some attributes are added in MG-IVHFIS.



(a) The multi-granulation space in MG-IVHFIS (b) The original upper approximation about $R_i^{A\widehat{\geq}}$ (c) The updated upper approximation after an addition (d) The specific variation

Fig. 5. The process of updating the upper approximation when some attributes are added in MG-IVHFIS.

4.3. The updating mechanism about adding some attributes

When some attributes are added to original MG-IVHFIS, the corresponding incremental mechanism is narrated in detail through this part. The attribute set which will be added is denoted by C. Equally, the variation of upper and lower approximations are marked by "$\top$" and "$\perp$".

Proposition 4.4. *Let $I^{\widehat{\geq}} = (U, A, V, f)$ be a MG-IVHFIS. For any $X \subseteq U$, we can update the lower and upper approximations of X as follows:*

(1) $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X) \cup \perp$, *where* $\perp = \{x \in ((X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)))| \vee_{i=1}^{s} ([x]_{R_i}^{A\cup C\widehat{\geq}} \subseteq X)\}$.

(2) $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = (\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - \top$, *where* $\top = \{x \in ((\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X)| \vee_{i=1}^{s} ([x]_{R_i}^{A\cup C\widehat{\geq}} \cap X = \emptyset)\}$.

(3) $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X) \cup \perp$, *where* $\perp = \{x \in ((X - \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)))| \wedge_{i=1}^{s} ([x]_{R_i}^{A\cup C\widehat{\geq}} \subseteq X)\}$.

(4) $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = (\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - \top$, *where* $\top = \{x \in ((\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X)| \wedge_{i=1}^{s} ([x]_{R_i}^{A\cup C\widehat{\geq}} \cap X = \emptyset)\}$.

Proof. Without loss of generality, we merely demonstrate properties under optimistic situation.

(1) Since it is evident that $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) \supseteq \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$ according to Proposition 4.2, there is a set $\perp$ making that $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X) \cup \perp$. That indicates we are going to examine objects, which belong to the lower approximation after inserting attribute set C. To reduce the time complexity, these retrieved objects come from $(X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X))$. Moreover, in certain granularity space, if there is an object x fulfilling that $[x]_{R_i}^{A\cup C\widehat{\geq}} \subseteq X$ after the addition, we will insert object x into the lower approximation. That is to say, $\perp = \{x \in ((X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)))| \vee_{i=1}^{s} ([x]_{R_i}^{A\cup C\widehat{\geq}} \subseteq X)\}$.

(2) What we know is that $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{B\cup C\widehat{\geq}}}(X) \subseteq \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{B\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$ based on Proposition 4.2. So it is similar that a set $\top$ plays a crucial role so that $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = (\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - \top$. This suggests objects which do not belong to $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$ after adding attribute set C are considered, namely objects belonging to $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$ should be examined. Here, object x comes from $(\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X$ to speed up the retrieval. Then, for s granularity spaces, object x is removed only if $[x]_{R_i}^{A\cup C\widehat{\geq}} \cap X = \emptyset$ in certain granularity space. That is to say, $\top = \{x \in ((\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X)| \vee_{i=1}^{s} ([x]_{R_i}^{A\cup C\widehat{\geq}} \cap X = \emptyset)\}$.

Algorithm 1: An algorithm for updating approximations of optimistic multi-granulation about deleting some attributes in dynamic MG-IVHFIS

Input:

(1) The original MG-IVHFIS $I^{\widehat{\geq}} = (U, A, V, f)$= {IVHFIS$_1$, IVHFIS$_2$, $\cdots$, IVHFIS$_s$} and a target set $X \subseteq U$.

(2) The original optimistic MG-IVHF lower and upper approximations of X: $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$.

(3) The attribute subset deleted from A: B.

Output: The new optimistic MG-IVHF lower and upper approximations of X after deleting attribute subset B: $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$.

begin
  Initialize: $\perp \leftarrow \emptyset$, $\top \leftarrow \emptyset$;
  for $i = 1 : s$ **do**
    Calculate $\{[x_1]_{R_i}^{A-B\widehat{\geq}}, [x_2]_{R_i}^{A-B\widehat{\geq}}, \cdots, [x_{|U|}]_{R_i}^{A-B\widehat{\geq}}\}$; // Compute average dominance classes based on $R_i^{A-B\widehat{\geq}}$.
  end
  for $x \in \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$ **do**
    for $i = 1 : s$ **do**
      if $[x]_{R_i}^{A-B\widehat{\geq}} \subseteq X$ **then**
        mark $\leftarrow 0$;
        break;
      else
        mark $\leftarrow 1$;
      end
    end
    if *mark = 1* **then**
      $\perp \leftarrow \perp \cup \{x\}$;
    end
  end
  $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X) \leftarrow \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) - \perp$; // Update the optimistic lower approximation of X.
  for $x \in (U - \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X))$ **do**
    for $i = 1 : s$ **do**
      if $[x]_{R_i}^{A-B\widehat{\geq}} \cap X = \emptyset$ **then**
        mark $\leftarrow 0$;
        break;
      else
        mark $\leftarrow 1$;
      end
    end
    if *mark = 1* **then**
      $\top \leftarrow \top \cup \{x\}$;
    end
  end
  $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X) \leftarrow \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \top$; // Update the optimistic upper approximation of X.
  return $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$;
end

Algorithm 2: An algorithm for updating approximations of pessimistic multi-granulation about deleting some attributes in dynamic MG-IVHFIS

Input:

(1) The original MG-IVHFIS $I^{\widehat{\geq}} = (U, A, V, f)$= {IVHFIS$_1$, IVHFIS$_2$, $\cdots$, IVHFIS$_s$} and a target set $X \subseteq U$.

(2) The original pessimistic MG-IVHF lower and upper approximations of X: $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$.

(3) The attribute subset deleted from A: B.

Output: The new pessimistic MG-IVHF lower and upper approximations of X after deleting attribute subset B: $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$.

begin
  Initialize: $\perp \leftarrow \emptyset$, $\top \leftarrow \emptyset$;
  for $i = 1 : s$ **do**
    Calculate $\{[x_1]_{R_i}^{A-B\widehat{\geq}}, [x_2]_{R_i}^{A-B\widehat{\geq}}, \cdots, [x_{|U|}]_{R_i}^{A-B\widehat{\geq}}\}$; // Compute average dominance classes based on $R_i^{A-B\widehat{\geq}}$.
  end
  for $x \in \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$ **do**
    for $i = 1 : s$ **do**
      if $[x]_{R_i}^{A-B\widehat{\geq}} \not\subseteq X$ **then**
        mark $\leftarrow 1$;
        break;
      else
        mark $\leftarrow 0$;
      end
    end
    if *mark = 1* **then**
      $\perp \leftarrow \perp \cup \{x\}$;
    end
  end
  $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X) \leftarrow \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) - \perp$; // Update the pessimistic lower approximation of X.
  for $x \in (U - \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X))$ **do**
    for $i = 1 : s$ **do**
      if $[x]_{R_i}^{A-B\widehat{\geq}} \cap X \neq \emptyset$ **then**
        mark $\leftarrow 1$;
        break;
      else
        mark $\leftarrow 0$;
      end
    end
    if *mark = 1* **then**
      $\top \leftarrow \top \cup \{x\}$;
    end
  end
  $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X) \leftarrow \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \top$; // Update the pessimistic upper approximation of X.
  return $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A-B\widehat{\geq}}}(X)$;
end

Table 5
The time complexity of Algorithm 3 and Algorithm 4.

Steps	*Time complexity*
2–6	$O(\lvert s\rvert(\lvert U\rvert^2 + \lvert U\rvert))$
7–19	$O(\lvert s\rvert\lvert(X - \underline{M}^{O(P)}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{O(P)}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X))\rvert)$
20	$O(1)$
21–33	$O(\lvert s\rvert\lvert(\overline{M}^{O(P)}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O(P)}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X\rvert)$
34	$O(1)$
Total	$O(\lvert s\rvert(\lvert U\rvert^2 + \lvert U\rvert))$

Table 6
The detailed description of data sets.

No.	Data sets	Abbreviation	Objects	Attributes
1	*Wine*	*W*	178	14
2	*Leaf*	*L*	340	16
3	*Indian liver patient*	*ILP*	583	10
4	*HCV data*	*HCV*	615	13
5	*Energy efficiency*	*EE*	768	10
6	*Concrete compressive strength*	*CCS*	1030	9
7	*Contraceptive Method Choice*	*CMC*	1473	10
8	*Wireless indoor localization*	*WIL*	2000	8
9	*Wine quality white*	*WQW*	4898	12
10	*Page blocks*	*PB*	5473	10

The process of updating approximations when some attributes are added in MG-IVHFIS is presented by means of Figs. 4 and 5. The variation of lower approximation can be watched in Fig. 4, among which (a) stands for the multi-granulation space, (b) and (c) reveal the original and updated lower approximations. The orange part of (d) represents these objects inserted into the original lower approximation, namely the set $\perp$. It is noticeable that the updated lower approximation becomes larger after an addition. In Fig. 5, the variation of lower approximation is displayed. Obviously, (a) represents the multi-granulation space, (b) and (c) reveal the upper approximation concerning $R_i^{A\widehat{\geq}}$ and $R_i^{A\cup C\widehat{\geq}}$. For (d), these objects removed from the original lower approximation are drawn in orange, namely the set $\top$. That means the updated upper approximation becomes smaller after an addition.

Algorithm 3 which is used for updating approximations with the addition of some attributes is designed based on above analysis. Similar to Algorithm 1, Algorithm 3 is also divided into three parts. In steps 2–6, we initialize $\perp$, $\top$ and calculate average dominance classes with regard to C and $A \cup C$. Steps 7–20 update the optimistic lower approximation of X, among which steps 9–14 make a judgment whether there is an arbitrary granularity satisfying that $[x]_{R_i}^{A\cup C\widehat{\geq}} \subseteq X$. In steps 21–34, the optimistic upper approximation of X is updated, among which steps 23–28 judge whether $[x]_{R_i}^{A\cup C\widehat{\geq}} \cap X \neq \emptyset$ for each granularity. In the end, new optimistic MG-IVHF lower and upper approximations are output. Likewise, the incremental algorithm of pessimistic multi-granulation is given as Algorithm 4. The time complexity of Algorithm 3 and Algorithm 4 are shown in Table 5.

5. Experimental analysis

In this section, a series of experiments are designed by us to verify the feasibility and effectiveness of our four dynamic algorithms. Ten data sets are downloaded by us from UCI Machine Learning Repository and details are available in Table 6. For some non-numerical data in several data sets such as time data and textual data, they are processed with the way of erasing time data and partial textual data that are difficult to be converted into numerical data, and converting other textual data into numerical data. All the experimental programs are executed on a computer with an Intel Core i7-9750H at 2.60 GHz, 8 GB RAM and Windows 10 (64-bit). These algorithms are accomplished by Python using an environment of Anaconda Navigator. In the experiment, the computation time is regarded as main evaluation index aiming to compare the dynamic method with the classical method.

What cannot be ignored is that the majority of data sets are made up of real numbers. To keep the experiment going, it is indispensable for us to construct MG-IVHFIS through a few steps. For simplicity, there are two granularities in the multi-granulation space and the value of object x under attribute a $f_a(x)$ contains two interval numbers at most for every $x \in U$ and $a \in A$ in MG-IVHFIS. Firstly, we normalize the selected data sets so that all the values between 0 and 1. Secondly, we adopt the measure of adding random noise into the original data sets to construct an information system with two granularities. For the second granularity, 60% of the data from original data set will be selected to add random noise while 40% of the data will remain unchanged. We mark 60% of the data that is selected randomly as set G. The random noise is added as follows:

$$f_a{}'(x) = \begin{cases} f_a(x) + r, & 0 \leq r \leq 1 \textit{ and } x \in G, \\ f_a(x), & x \notin G, \end{cases} \tag{15}$$

where $f_a{}'(x)$ signifies the value of object x under attribute a for the second granularity, r signifies the random noise in the range from 0 to 1. Thirdly, we may generate two interval numbers from a real number by calculating the formula $\{[(1-\psi) \times f_a(x), (1-\omega) \times f_a(x)], [(1+\omega) \times f_a(x), (1+\psi) \times f_a(x)]\}$. In this experiment, two error precisions ψ and ω are set as: $\psi \in [0.1, 0.15]$ and $\omega \in [0, 0.05]$. Finally, 35% of objects will be chosen randomly as the target set X for each data set, so as to accurately describe the target set through lower and upper approximations. Particularly, if $35\% \times |U|$ is not an integer, we will apply the floor function, namely $|X| = \lfloor 35\% \times |U| \rfloor$. Given that the single test result of data set divided by attributes is not stable enough and possibly inflicts errors, we have taken proactive steps. Aimed at assuring the balance property of data set which has been divided, the same process is adopted to reconstruct each data set. At the same time, during the course of division, random partition was conducted for multiple times, and the average value was taken after repeated experiments. In order to minimize the experimental error, we will carry out the program five times to obtain the average computation time as the final result.

5.1. The experiment with the deletion of some attributes

The first experiment has to do with removing attributes in MG-IVHFIS. For each data set, a certain percentage of attributes will be deleted each time, starting at 5% and ending at 50% of all attributes, increasing by 5% at a time up to 10 deletions. Two points that need attention are as described: at first, these deleted attributes are the same every time under two granularities and are the same for dynamic and static algorithms. Besides, when the magnitude of deleted attributes is not an integer, the floor function is still adopted.

The final experimental results are presented in Tables 7 and 8, where the unit of computation time is seconds. Dyn. and Cla. are short for dynamic and classical algorithms respectively. Aimed at making the comparison about the effect of deleting attributes between the dynamic and classical methods more legible and intuitive, we draw some associative three-dimensional surfaces in accordance with Tables 7 and 8, and detailed information is displayed in Fig. 6. With regard to each subgraph in Fig. 6, the x-coordinate represents the quantity ratio of deleted attributes to original attributes, the y-coordinate indicates four methods, namely classical and dynamic approaches in the optimistic and

Algorithm 3: An algorithm for updating approximations of optimistic multi-granulation about adding some attributes in dynamic MG-IVHFIS

Input:

(1) The original MG-IVHFIS $I^{\widehat{\geq}} = (U, A, V, f) = \{\mathrm{IVHFIS}_1, \mathrm{IVHFIS}_2, \cdots, \mathrm{IVHFIS}_s\}$, a target set $X \subseteq U$, and $\{[x_1]_{R_i}^{A-C\widehat{\geq}}, [x_2]_{R_i}^{A-C\widehat{\geq}}, \cdots, [x_{|U|}]_{R_i}^{A-C\widehat{\geq}}\}$ $(i = 1, 2, \cdots, s)$.

(2) The original optimistic MG-IVHF lower and upper approximations of X: $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$.

(3) The attribute set added to A: C.

Output: The new optimistic MG-IVHF lower and upper approximations of X after adding attribute set C: $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$.

begin
  Initialize: $\perp \leftarrow \emptyset$, $\top \leftarrow \emptyset$;
  for $i = 1 : s$ **do**
    Calculate $\{[x_1]_{R_i}^{C\widehat{\geq}}, [x_2]_{R_i}^{C\widehat{\geq}}, \cdots, [x_{|U|}]_{R_i}^{C\widehat{\geq}}\}$ and $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$;
    Calculate $[x]_{R_i}^{A\cup C\widehat{\geq}} \leftarrow [x]_{R_i}^{A\widehat{\geq}} \cap [x]_{R_i}^{C\widehat{\geq}}$; // Compute average dominance classes based on $R_i^{A\cup C\widehat{\geq}}$.
  end
  for $x \in ((X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)))$ **do**
    for $i = 1 : s$ **do**
      if $[x]_{R_i}^{A\cup C\widehat{\geq}} \subseteq X$ **then**
        mark $\leftarrow 1$;
        break;
      else
        mark $\leftarrow 0$;
      end
    end
    if *mark = 1* **then**
      $\perp \leftarrow \perp \cup \{x\}$;
    end
  end
  $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \underline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X) \cup \perp$; // Update the optimistic lower approximation of X.
  for $x \in ((\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X)$ **do**
    for $i = 1 : s$ **do**
      if $[x]_{R_i}^{A\cup C\widehat{\geq}} \cap X = \emptyset$ **then**
        mark $\leftarrow 1$;
        break;
      else
        mark $\leftarrow 0$;
      end
    end
    if *mark = 1* **then**
      $\top \leftarrow \top \cup \{x\}$;
    end
  end
  $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = (\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{O}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - \top$; // Update the optimistic upper approximation of X.
  return $\underline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$, $\overline{M}^{O}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$;
end

pessimistic environment. Meanwhile the z-coordinate shows the computation time of four methods. As can be observed in these subgraphs, it is distinct to discover that with the quantity of deleted attributes growing, the computation time of classical

Algorithm 4: An algorithm for updating approximations of pessimistic multi-granulation about adding some attributes in dynamic MG-IVHFIS

Input:

(1) The original MG-IVHFIS $I^{\widehat{\geq}} = (U, A, V, f)$= {IVHFIS$_1$, IVHFIS$_2$, $\cdots$, IVHFIS$_s$}, a target set $X \subseteq U$, and $\{[x_1]_{R_i}^{A-C\widehat{\geq}}, [x_2]_{R_i}^{A-C\widehat{\geq}}, \cdots, [x_{|U|}]_{R_i}^{A-C\widehat{\geq}}\}$ $(i = 1, 2, \cdots, s)$.

(2) The original pessimistic MG-IVHF lower and upper approximations of X: $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)$.

(3) The attribute set added to A: C.

Output: The new pessimistic MG-IVHF lower and upper approximations of X after adding attribute set C: $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$.

begin
　Initialize: $\perp \leftarrow \emptyset$, $\top \leftarrow \emptyset$;
　for $i = 1 : s$ **do**
　　Calculate $\{[x_1]_{R_i}^{C\widehat{\geq}}, [x_2]_{R_i}^{C\widehat{\geq}}, \cdots, [x_{|U|}]_{R_i}^{C\widehat{\geq}}\}$ and $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)$;
　　Calculate $[x]_{R_i}^{A\cup C\widehat{\geq}} \leftarrow [x]_{R_i}^{A\widehat{\geq}} \cap [x]_{R_i}^{C\widehat{\geq}}$;　// Compute average dominance classes based on $R_i^{A\cup C\widehat{\geq}}$.
　end
　for $x \in ((X - \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X)) \cap (X - \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)))$ **do**
　　for $i = 1 : s$ **do**
　　　if $[x]_{R_i}^{A\cup C\widehat{\geq}} \not\subseteq X$ **then**
　　　　mark $\leftarrow 0$;
　　　　break;
　　　else
　　　　mark $\leftarrow 1$;
　　　end
　　end
　　if *mark = 1* **then**
　　　$\perp \leftarrow \perp \cup \{x\}$;
　　end
　end
　$\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cup \underline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X) \cup \perp$;　// Update the pessimistic lower approximation of X.
　for $x \in ((\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - X)$ **do**
　　for $i = 1 : s$ **do**
　　　if $[x]_{R_i}^{A\cup C\widehat{\geq}} \cap X \neq \emptyset$ **then**
　　　　mark $\leftarrow 0$;
　　　　break;
　　　else
　　　　mark $\leftarrow 1$;
　　　end
　　end
　　if *mark = 1* **then**
　　　$\top \leftarrow \top \cup \{x\}$;
　　end
　end
　$\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X) = (\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\widehat{\geq}}}(X) \cap \overline{M}^{P}_{\sum_{i=1}^{s} R_i^{C\widehat{\geq}}}(X)) - \top$;　// Update the pessimistic upper approximation of X.
　return $\underline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$, $\overline{M}^{P}_{\sum_{i=1}^{s} R_i^{A\cup C\widehat{\geq}}}(X)$;
end

method presents a gradual decreasing tendency, while the computation time of dynamic method fluctuates relatively little. What we may suppose is that if the amount of deleted attributes is massive enough, the efficiency of classical and dynamic algo-

Table 7
The comparison of computation time between classical algorithm and dynamic algorithm with a certain ratio of deleting attributes about optimistic multi-granulation (OM)

Ratio	Method	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
W	Cla.	0.415	0.399	0.370	0.362	0.334	0.328	0.316	0.252	0.227	0.216
	Dyn.	0.072	0.069	0.070	0.071	0.071	0.073	0.073	0.071	0.070	0.070
L	Cla.	1.927	1.808	1.737	1.587	1.511	1.363	1.280	1.249	1.119	1.030
	Dyn.	0.282	0.265	0.266	0.264	0.261	0.265	0.281	0.272	0.283	0.266
ILP	Cla.	4.076	3.818	3.666	3.518	3.218	3.306	2.898	2.446	2.233	2.175
	Dyn.	0.764	0.807	0.802	0.789	0.795	0.808	0.814	0.830	0.725	0.782
HCV	Cla.	5.308	4.846	4.712	4.414	4.031	3.613	3.606	3.390	3.065	2.717
	Dyn.	0.925	0.924	0.968	0.926	0.955	0.905	0.946	0.975	0.988	0.964
EE	Cla.	5.965	5.759	5.450	4.893	5.020	4.587	4.454	4.360	3.855	3.346
	Dyn.	1.246	1.196	1.182	1.280	1.224	1.306	1.278	1.306	1.255	1.249
CCS	Cla.	9.992	9.944	9.310	9.046	8.490	7.809	7.165	7.198	6.189	5.521
	Dyn.	2.199	2.241	2.360	2.298	2.355	2.383	2.358	2.385	2.205	2.311
CMC	Cla.	22.137	20.719	20.322	18.969	17.968	17.379	17.738	16.245	15.367	14.009
	Dyn.	5.389	5.485	5.564	5.417	5.705	5.505	5.540	5.614	5.391	5.241
WIL	Cla.	35.415	34.353	34.231	30.865	29.203	29.339	28.664	25.965	23.150	21.156
	Dyn.	8.275	8.997	8.967	8.777	9.101	8.882	7.812	8.522	8.365	7.899
WQW	Cla.	324.154	292.676	289.206	282.934	237.820	244.327	216.831	198.157	173.286	171.665
	Dyn.	51.844	52.429	52.196	52.685	50.982	50.665	49.891	51.945	48.265	50.220
PB	Cla.	343.340	313.708	313.171	290.875	293.123	273.379	267.116	231.282	204.694	174.508
	Dyn.	75.063	77.711	76.286	73.557	75.329	76.099	72.866	78.135	76.132	70.011

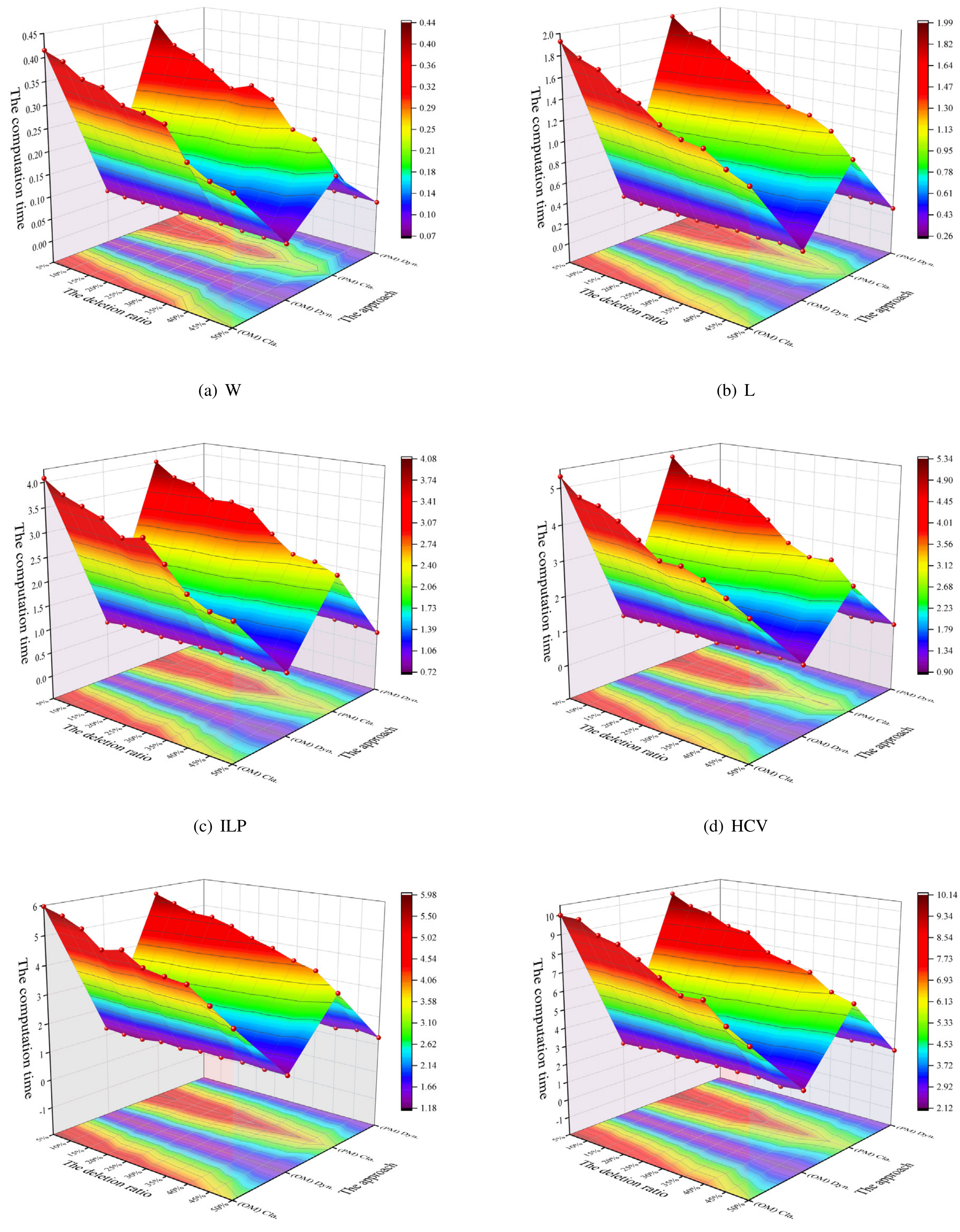
Table 8
The comparison of computation time between classical algorithm and dynamic algorithm with a certain ratio of deleting attributes about pessimistic multi-granulation (PM)

Ratio	Method	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
W	Cla.	0.434	0.387	0.371	0.344	0.312	0.326	0.304	0.248	0.237	0.169
	Dyn.	0.074	0.066	0.071	0.068	0.073	0.073	0.069	0.072	0.071	0.070
L	Cla.	1.987	1.838	1.790	1.655	1.552	1.399	1.291	1.250	1.141	0.921
	Dyn.	0.271	0.275	0.269	0.264	0.265	0.267	0.273	0.280	0.275	0.269
ILP	Cla.	4.021	3.738	3.660	3.393	3.415	3.318	2.896	2.564	2.502	2.324
	Dyn.	0.810	0.854	0.833	0.849	0.815	0.836	0.823	0.829	0.799	0.765
HCV	Cla.	5.333	4.813	4.775	4.600	4.402	3.943	3.394	3.109	3.157	2.567
	Dyn.	0.906	0.944	0.928	0.921	1.017	0.955	0.976	0.902	0.912	0.965
EE	Cla.	5.758	5.507	5.266	5.214	5.003	4.698	4.472	4.173	3.963	3.353
	Dyn.	1.210	1.203	1.261	1.277	1.334	1.239	1.305	1.267	1.368	1.254
CCS	Cla.	10.123	9.566	9.312	8.770	8.587	7.683	7.356	7.024	6.193	5.821
	Dyn.	2.131	2.309	2.253	2.478	2.435	2.351	2.369	2.384	2.456	2.356
CMC	Cla.	22.361	21.307	21.055	19.506	18.958	17.646	18.106	15.717	14.814	12.972
	Dyn.	5.315	5.597	5.617	5.640	5.573	5.373	5.594	5.312	5.334	5.296
WIL	Cla.	35.198	33.476	33.080	31.487	30.234	30.464	29.239	25.881	25.572	22.409
	Dyn.	8.384	8.728	8.867	9.299	8.999	9.348	8.650	8.817	8.589	8.682
WQW	Cla.	322.761	306.909	270.790	265.658	243.572	233.938	210.701	210.021	182.587	169.323
	Dyn.	49.412	56.811	52.904	51.123	48.293	49.059	49.828	49.878	51.836	48.598
PB	Cla.	344.093	321.347	308.259	279.666	268.763	263.075	251.110	226.094	195.435	171.898
	Dyn.	73.540	75.331	72.998	70.219	72.875	71.296	70.745	72.684	67.867	61.677

rithms will be closer and closer. Moreover, when the amount of deleted attributes is no more than half of the original attributes, the computation time of dynamic method is significantly lower than that of classical method. The cardinality of object set and the cardinality of attribute set in a data set are pivotal factors that impact on the performance of four methods. To sum up, the dynamic updating approximations approach about removing attributes is much swifter than classical approach in the first experiment.

5.2. The experiment with the addition of some attributes

The second experiment has to do with inserting attributes in MG-IVHFIS. Under different granularities, attribute set for each data set is split, in which 60% of attribute set is picked out as initial attribute set, and remanent 40% of attribute set is deemed as the attribute set to be added. To make our illustration more concise, 40% of original attributes which is to be inserted are recorded as W. A certain percentage of attributes which are in



(a) W (b) L

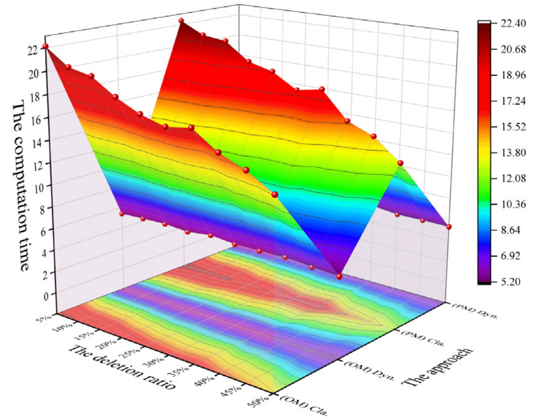
(c) ILP (d) HCV

(e) EE (f) CCS

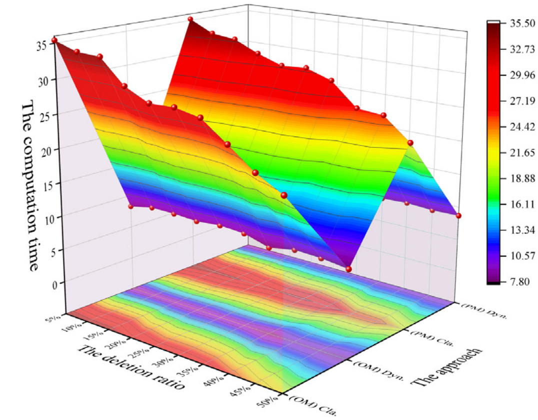
Fig. 6. Line charts of computation time about classical and dynamic methods with a certain ratio of deleting attributes.

W will be added to the initial attribute set each time, starting at 10% and ending at 100% of W, increasing by 10% at a time up to 10 additions. Considerations resemble the first experiment: under two granularities and for two approaches, these added attributes are identical every time. Equally, the floor function will play a role if the magnitude of added attributes is not an integer.

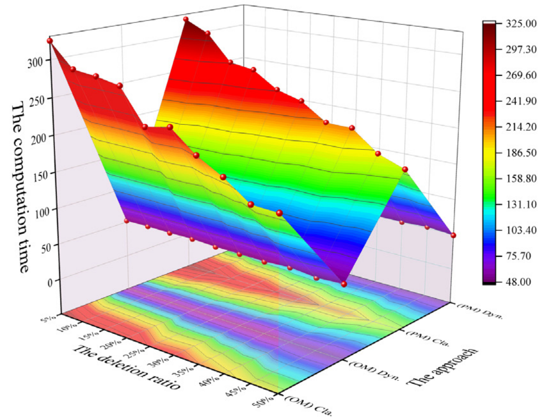
Having access to dynamic and classical methods, the time that has been consumed to add attributes is shown in Tables 9 and 10. Then, we plot the time in the form of three-dimensional surfaces as exhibited in Fig. 7. Determined by the computation time in Tables 9 and 10, Fig. 7 consists of 10 subgraphs as well, where the x-coordinate stands for the quantity ratio of added attributes to W, the y-coordinate stands for classical and dynamic approaches from the viewpoint of optimistic and pessimistic multi-granulation, and the z-coordinate stands for the computation time. From general directions of these subgraphs,



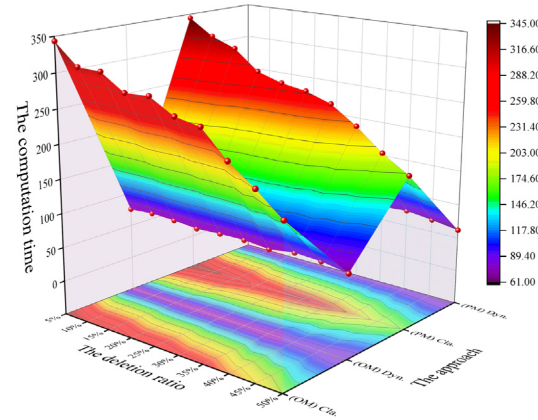
(g) CMC



(h) WIL



(i) WQW



(j) PB

Fig. 6. (*continued*).

Table 9
The comparison of computation time between classical algorithm and dynamic algorithm with a certain ratio of adding attributes about optimistic multi-granulation (OM)

Ratio	Method	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
W	Cla.	0.194	0.217	0.246	0.286	0.318	0.322	0.352	0.378	0.410	0.439
	Dyn.	0.010	0.038	0.071	0.098	0.133	0.135	0.161	0.193	0.231	0.265
L	Cla.	0.826	0.961	1.097	1.234	1.326	1.452	1.589	1.726	1.869	2.026
	Dyn.	0.037	0.146	0.270	0.396	0.539	0.611	0.749	0.875	1.053	1.171
ILP	Cla.	1.849	2.247	2.289	2.697	3.066	3.106	3.448	3.536	3.857	4.191
	Dyn.	0.114	0.450	0.478	0.812	1.240	1.251	1.617	1.652	1.973	2.359
HCV	Cla.	2.461	2.896	3.391	3.379	3.772	4.224	4.305	4.856	5.149	5.597
	Dyn.	0.125	0.498	0.894	0.941	1.350	1.752	1.786	2.206	2.653	3.062
EE	Cla.	2.611	3.347	3.372	3.930	4.638	4.655	5.235	5.242	6.147	6.859
	Dyn.	0.200	0.808	0.785	1.424	2.027	2.078	2.762	2.730	3.348	4.119
CCS	Cla.	4.892	5.899	6.015	7.210	7.196	8.551	8.787	10.001	10.038	11.101
	Dyn.	0.361	1.392	1.414	2.578	2.605	3.775	3.792	5.159	5.197	6.322
CMC	Cla.	9.571	12.359	12.304	14.743	16.823	17.060	19.558	19.559	21.989	24.401
	Dyn.	0.863	2.948	2.893	5.148	7.360	7.427	9.761	9.655	12.509	14.541
WIL	Cla.	17.516	18.063	22.712	23.290	27.355	27.825	27.360	31.650	31.167	34.941
	Dyn.	1.487	1.547	5.655	5.407	9.663	9.982	9.658	13.921	14.127	18.024
WQW	Cla.	147.857	176.573	203.151	199.544	231.924	248.289	250.165	271.948	294.121	316.064
	Dyn.	13.237	59.105	81.658	86.162	100.581	121.571	123.794	151.875	166.890	191.870
PB	Cla.	153.102	182.454	185.040	211.328	238.351	253.385	275.100	280.170	302.006	326.801
	Dyn.	19.579	63.045	70.133	103.368	132.031	137.741	164.501	165.585	194.546	219.328

Table 10
The comparison of computation time between classical algorithm and dynamic algorithm with a certain ratio of adding attributes about pessimistic multi-granulation (PM)

Ratio	Method	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
W	Cla.	0.199	0.224	0.256	0.289	0.314	0.321	0.352	0.391	0.422	0.460
	Dyn.	0.012	0.038	0.072	0.099	0.127	0.130	0.165	0.192	0.223	0.263
L	Cla.	0.850	0.989	1.095	1.236	1.403	1.523	1.649	1.772	1.919	2.114
	Dyn.	0.038	0.150	0.273	0.406	0.509	0.650	0.801	0.900	1.045	1.161
ILP	Cla.	1.791	2.176	2.257	2.657	3.085	3.118	3.477	3.502	3.874	4.359
	Dyn.	0.112	0.449	0.501	0.815	1.209	1.213	1.583	1.644	1.985	2.382
HCV	Cla.	2.509	2.908	3.346	3.430	3.960	4.421	4.472	4.890	5.182	5.623
	Dyn.	0.125	0.499	0.929	0.964	1.400	1.851	1.873	2.233	2.706	3.076
EE	Cla.	2.514	3.221	3.257	3.841	4.548	4.611	5.163	5.196	5.859	6.498
	Dyn.	0.209	0.799	0.774	1.376	2.028	2.083	2.645	2.680	3.321	3.889
CCS	Cla.	4.716	5.851	5.993	7.325	7.205	8.458	8.531	9.923	9.952	10.929
	Dyn.	0.363	1.368	1.423	2.583	2.516	3.734	3.760	5.188	5.215	6.469
CMC	Cla.	9.684	12.099	12.179	14.785	17.305	17.264	19.558	19.353	22.408	24.769
	Dyn.	0.842	2.981	2.883	5.138	7.645	7.702	10.113	10.021	12.644	15.008
WIL	Cla.	18.771	19.104	23.783	23.716	28.383	28.548	28.832	33.087	32.431	37.068
	Dyn.	1.435	1.466	5.599	5.779	10.118	10.303	10.186	14.569	14.449	18.899
WQW	Cla.	152.517	175.124	200.607	198.990	225.105	248.533	250.436	269.672	286.348	325.128
	Dyn.	16.720	50.997	75.586	91.007	104.307	130.052	127.254	153.859	175.548	198.744
PB	Cla.	156.636	181.989	183.468	210.383	247.372	260.638	284.067	289.131	302.421	330.412
	Dyn.	27.187	67.430	79.195	103.457	135.086	143.324	158.217	158.008	195.546	212.968

we may reach an agreement: as the quantity of added attributes augments, the computation time of both classical and dynamic methods tends to increase gradually. Given that two curves about classical and dynamic methods under the condition of optimism or pessimism have no tendency to intersect, we infer that if the amount of attributes inserted into a data set is massive enough, the dynamic method will maintain significant superiority. After contrasting the z-coordinate among various subgraphs, a conclusion similar to the first experiment is that the larger the cardinality of object set and attribute set is, the longer the time of updating approximations is. Generally speaking, dynamic approach is always better than classical approach when it comes to inserting attributes.

6. Conclusions

Dynamic updating approximations method is an ingenious strategy in data mining and knowledge discovery. It is the mechanism of acquiring the latest knowledge on basis of previous knowledge in a time-evolving information system that makes dynamic updating approximations extremely high-efficiency. In this paper, we go over essential notions of multi-granulation interval-valued hesitant fuzzy rough set, as well as some concepts of information systems and dominance relation. In reality, the conventional dominance relation is rigorous for practical issue. Therefore, we propose the average dominance relation based on dominance degree, then construct a RS model which relies on the average dominance relation in MG-IVHFIS. In addition, four dynamic mechanisms associated with attributes changing while objects hold constant have been investigated in MG-IVHFIS, including two situations of deleting attributes and adding attributes. Finally, in order to compare the efficiency of dynamic and classical methods, a series of experiments concerning ten UCI data sets are implemented to certify the effectiveness of dynamic algorithm. The experimental results indicate that dynamic approaches lessen the time consumption and accelerate the computational efficiency when attributes vary in MG-IVHFIS.

In this paper, only updating approximations of MG-IVHFIS are explored. It is proved that these incremental algorithms significantly shorten the computation time. Nonetheless, since the variation is normally multidimensional in an information system, the reality is more intricate. Confronted with plentiful data and multifarious problems, our future work will extend dynamic algorithms and apply them to MG-IVHFIS in which multidimensional variation arises. Furthermore, we are committed to further improving the efficiency of dynamic approaches and continuing to use these mechanisms in more information systems. Additionally, we intend to link the incremental mechanism to attribute reduction in an effort to accelerate the speed of attribute reduction.

CRediT authorship contribution statement

Xiaoyan Zhang: Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation. **Jirong Li:** Data curation, Methodology, Software, Visualization, Writing-Original draft preparation, Writing-Reviewing & Editing. **Jusheng Mi:** Investigation, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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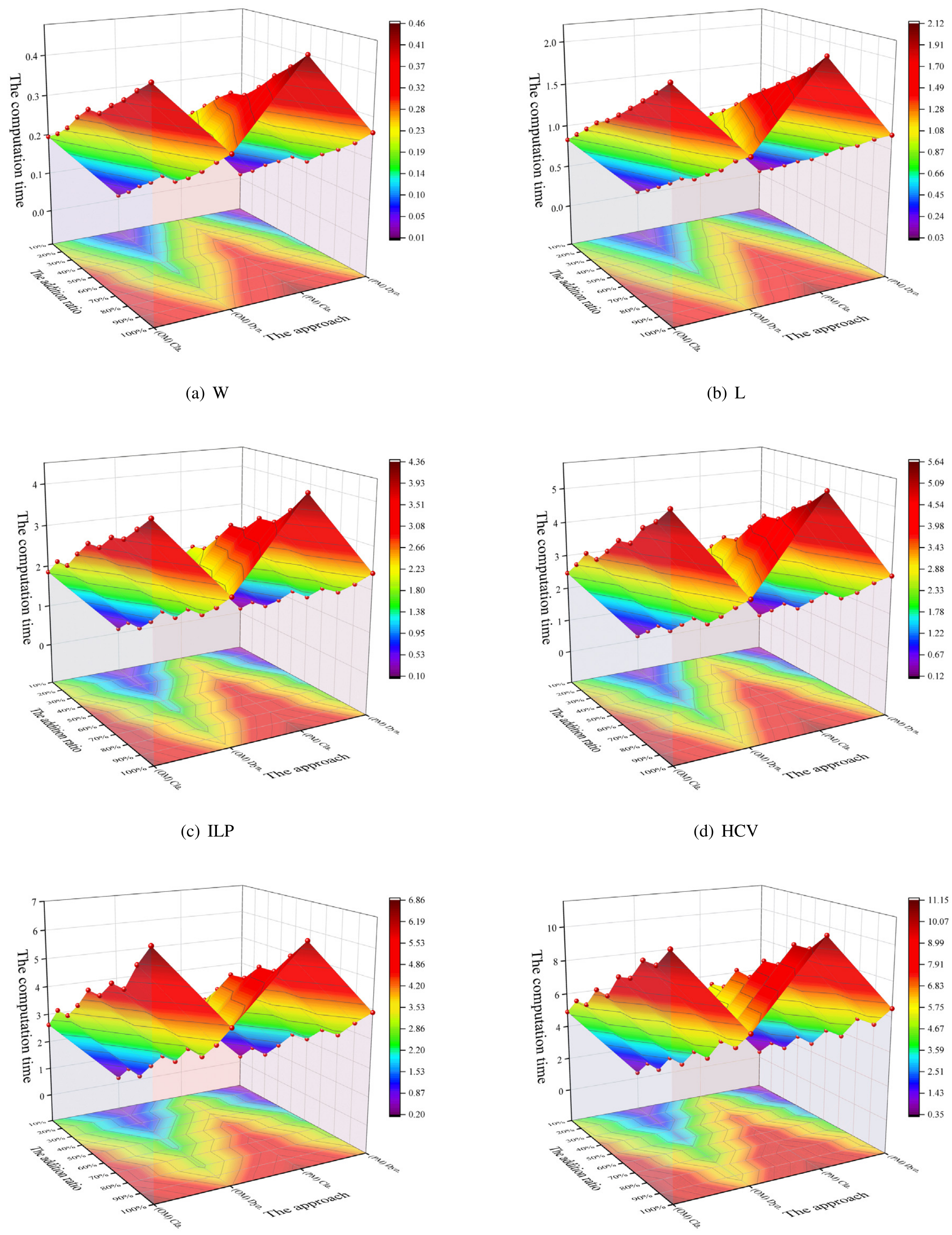
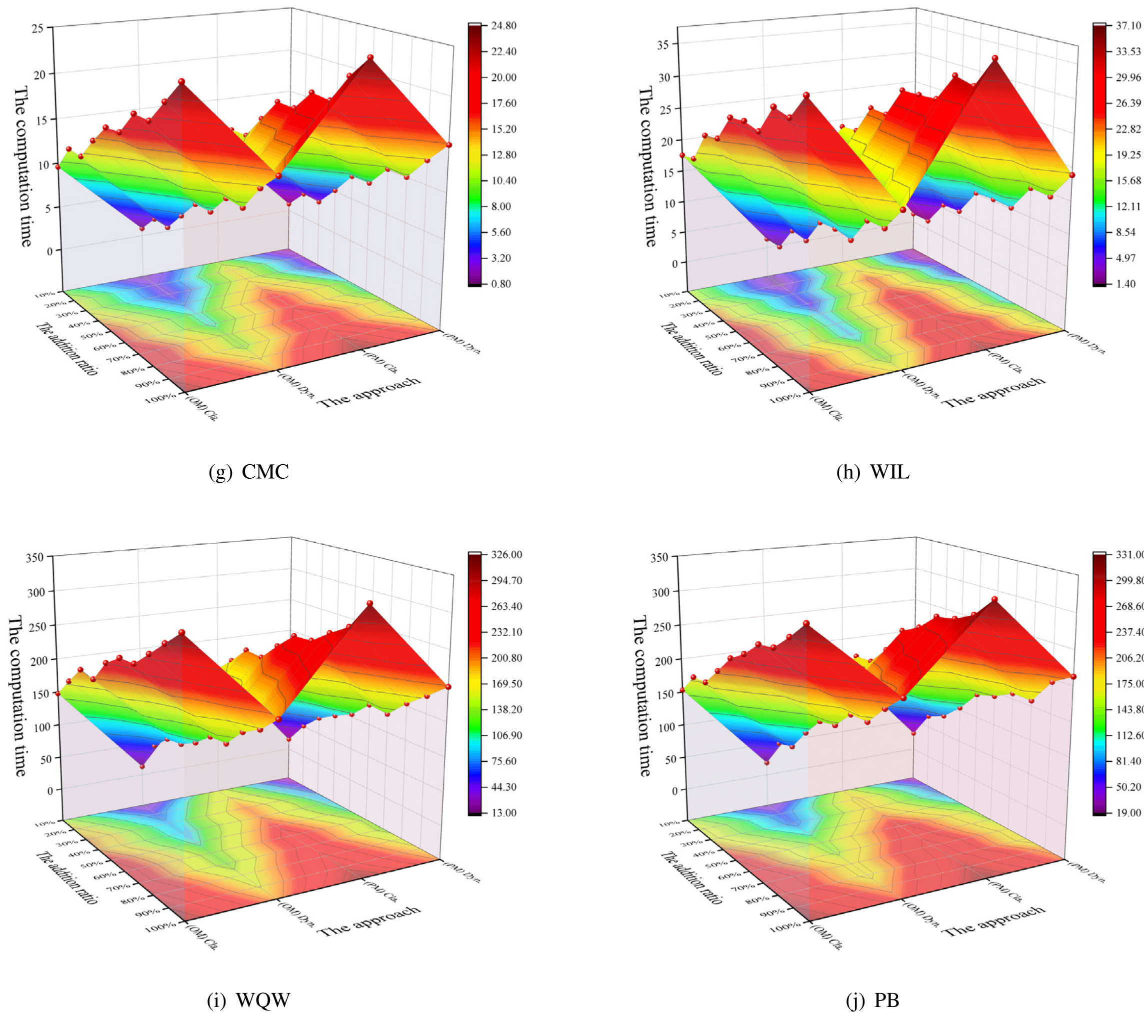


(a) W (b) L

(c) ILP (d) HCV

(e) EE (f) CCS

Fig. 7. Line charts of computation time about classical and dynamic methods with a certain ratio of adding attributes.



(g) CMC (h) WIL

(i) WQW (j) PB

Fig. 7. (*continued*).

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